NOTES ON THE HISTORY OF THE GENERAL EQUATIONS OF HYDRODYNAMICS*

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Introduction. The following article consists mainly in the descriptions of volumes in an exhibit in the Indiana University Mathematics Library commemorating the two-hundredth anniversary of the general equations of fluid dynamics.

1752–1952. In 1748 the Berlin Academy proposed as the subject for the prize competition of 1750 the theory of the resistance of fluids. The tentatives of Newton were known to be wrong, and nothing but empirical formulae had taken their place. The subject, besides being close to the unexplored foundations of the concept of inertia, was thus of great mathematical interest and practical moment.

In December, 1749, d'Alembert sent to Berlin his entry, a long *Essai d'une Nouvelle Théorie de la Résistance des Fluides*. The Academy, of which d'Alembert

* There are only two relatively extensive historical works on fluid dynamics:

O. Flachsbart, Geschichte der experimentellen Hydro- und Aeromechanik, insbesondere der Widerstandsforschung, pp. 1–61 of vol. 4$_2$ of *Handbuch der Experimentalphysik*, Leipzig, 1932.

R. Giacomelli & E. Pistolesi, Historical Sketch (of aerodynamics), pp. 305–394 in vol. 1 of *Aerodynamic Theory*, ed. W. F. Durand, Berlin, 1934.

Neither is complete nor entirely reliable; in both, the mathematical aspects of the subject are almost entirely neglected. There is some relevant material in

E. Hoppe, Geschichte der Physik, pp. 1–179 of vol. 1 *Handbuch der Phys.*, Berlin, 1926,

R. Dugas, *Histoire de la Mécanique*, Neuchâtel and Paris, 1950.

Both these works are scholarly; the former is entirely reliable, the latter, mainly so.

The few remarks concerning hydrodynamics in the standard histories of mathematics or of physics, when not entirely false, are so disconnected and incomplete as to be virtually meaningless. Many consist in more or less distorted selections from

Lagrange, *Méchanique Analitique*, Paris, 1788; see Seconde Partie, Septième Section. Slightly enlarged in the second and later editions (*e.g. Oeuvres* **11, 12**), where it appears as Section Dixième.

This brief and elegantly written history is scholarly, accurate in statement, and penetrating, but it omits most of the subject entirely. Lagrange dismisses all the early work with a phrase uniting Archimedes and Galilei: "for the interval which separated these two great geniuses disappears in the history of mechanics." It is only total ignorance which can produce such total contempt. The most serious omission of all, only partly corrected in the second edition, is the inexplicable neglect of Euler, from whose papers most of the portions of the *Mécanique Analytique* which concern fluids are taken over almost unchanged. And of course it stops in 1788.

My teacher, the late Dr. P. F. Neményi, left unfinished the MS of a remarkable history of the concepts of fluid mechanics, dealing mainly with the physical side of the subject. Within the next two years I hope to be able to fulfill our plan of balancing it by a corresponding study of the mathematical ideas and methods.

Meanwhile, however, I have thought it worthwhile to publish these remarks on the formation of the backbone of the whole subject. At the referee's suggestion, I have added this explanatory note and a few references to other historical works; in this connection it should perhaps be added that any statement for which no such reference is given represents my own opinion, formed entirely from study of the original sources.

was a non-resident member, judged that no contestant had earned the prize, sent back all manuscripts, and urged the authors to compare their predictions with the results of experiment. D'Alembert withdrew at once, and in 1752 he published his essay. It is described in §3 below. In the introduction d'Alembert details the circumstances in angry and sarcastic terms.

At the time of the contest Euler was director of the Mathematical Division of the Berlin Academy. It seems difficult to believe that he could have expected seriously that a theory of fluids could give results in accord with existing experiments on the resistance suffered by submerged bodies. The data itself was confused and sometimes contradictory. Nearly all the experiments concerned turbulent flow; they were not to be understood even qualitatively for a full century following; and to the present day the phenomenon of resistance remains, as far as any rigorous theory is concerned, a partially open question. Rather, it is likely that the reason given out was only a pretext, offered in place of the truth, much more difficult to substantiate to an author, that d'Alembert's reasoning was tortuous, incomprehensible, and endless, and that in illustration of his equations he had not succeeded in exhibiting a single two-dimensional flow.

This view is supported by the fact that by 1752 Euler had read to the Academy a paper *De motu fluidorum in genere*, treating this subject. Eneström* identified this paper with that described in §4 below. No resistances of specific bodies are calculated in it, nor in any of Euler's papers on the general theory, and there is no attempt to compare predictions with experimental measurements. But much that d'Alembert had tried to do is achieved in a direct and economical fashion, not only more general but also far more convincing—as Euler said in a later paper, "directly from the first axioms of mechanics."

It was the facile opinion of Euler's contemporaries, repeated in most of the present histories of mathematics, that† "M. Euler seemed sometimes to be occupied by the pure pleasure of mathematical calculation and of considering the questions of mechanics and physics only as occasions for exercising his genius and for abandoning himself to his predominant passion. So scientists have reproached him for having sometimes lavished his calculus on physical hypotheses, or even on metaphysical principles, of which he had not sufficiently examined the likelihood and solidity." In contrast, d'Alembert is represented as a physicist who "explained with his usual clearness and insight the necessary dependence of calculation upon experiment."‡ This estimate of d'Alembert seems to derive from Lagrange, who in various places in the *Mécanique Analytique* attributed to d'Alembert various things which it is extremely difficult to locate at all in the actual writings of d'Alembert himself, and extremely hard to follow even when located. I believe that any modern scientist who actually

* Verzeichnis der Schriften Leonhard Eulers, *Jahresbericht der deutschen Math. Verein.*, 4. Erganzungsband, 1910. See No. 258.

† Éloge de M. Euler (by Condorcet), *Hist. Acad. Sci. Paris*, 1783, (1786), quoted by Giacomelli and Pistolesi, *op. cit.*, p. 325.

‡ Giacomelli and Pistolesi, *loc. cit.*

reads the papers of d'Alembert and Euler will soon adopt a directly reversed view. In fact, in theories of nature Euler was so superior to all his contemporaries that parts of his writings on mechanics and physics were dismissed as too "mathematical," while to the modern reader they appear perfectly clear and in large part correct. It should be added also that Euler not only kept himself fully informed in experimental physics, but was always ready to make immediate use of the results of his calculations—and moreover, that those of his hydrodynamical writings which made the greatest permanent advances are those least influenced by experiment, while those in which he compromised with practicality have lost all but historical interest, since the experimental data available was obscure, incomplete, and sometimes faulty.*

The work of d'Alembert in hydrostatics was examined in detail by Todhunter, who found it "almost uniformly wrong" and containing "but little of importance." After careful search of his writings on hydrodynamics, I find the same words applicable, except for the fact that he did somehow arrive at the correct equations for plane and axially-symmetric irrotational flow. Philosophical meandering and doubts regarding the validity of the theory he is constructing fill so great a part of his work that there is little space left for development of the theory itself. It is quite possible that d'Alembert's discussion of experiments and repeated criticism of the mathematical method in physics may have appealed to some physicists of his day, as doubtless in ours; but no progress, either in mathematics or in physics, was achieved by these merely negative statements.

1. I. Newton, *Philosophiae naturalis principia mathematica*, London, 1687. Newton's views on fluids were heterodox. In the *Principia* we find both a molecular model of a "rare medium, consisting of equal particles freely disposed at equal distances from each other," and side by side with it a representation of a fluid as a smooth continuum without voids. The former view, in which impacts of the fluid particles upon each other and hence also those upon the back side of a submerged body are neglected, led Newton to the famous law of resistance proportional to the square of the velocity, and to his difficult and incorrect theory of sound. It has excited a great deal of notice ever since 1687, but has not led to any significant development in hydrodynamics.

One of the several passages where fluids are regarded as continuous media may be found in Lib. II, Sect. IX. Here Newton states a "Hypothesis," which may be translated as follows: "the resistance arising from the want of slipperiness in the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid separate from one another." It is the earliest correct statement regarding the internal friction of fluids. When properly generalized by Stokes (see §10), it forms the basis of the modern theory of

* A detailed analysis of Euler's writings on hydrodynamics will appear shortly in the introduction to vols. 12 and 13 of *L. Euleri Opera Omnia*, ser. 2.

viscosity. Theorem 39, however, which follows immediately, is false, as is virtually every specific conclusion of Newton concerning fluids.

Historical research has destroyed much of the attribution of originality to Book I of the *Principia*, where occurs much of the now standard material given in elementary mechanics courses. This part of the treatise now appears mainly as an attempt at organization and rigorous, deductive treatment of known material. Earlier works contained errors freely intermingled with correct results, while Newton's presentation is correct in principle and in detail. Book II, however, which concerns the properties of fluids and of bodies moving in them, consists almost entirely in original work of Newton, who was the founder of hydrodynamics. The plan of deducing everything from the axioms, so distinctive a feature of Book I, is abandoned, and fresh hypotheses are put forward at every turn. We may gather from a remark in the preface that it is necessity, not choice, which has enforced this breakdown of the original scheme. Newton's might of intellect is revealed even more clearly in this largely erroneous but path-breaking part of the work, which gave rise to two hundred years of controversy in the attempt to correct and complete what Newton had begun. Finally, in Book III Newton shows his peerless ability to get specific, numerical conclusions from an analytical theory and to compare them with the results of experiment.

2. D. Bernoulli, "Theoria nova de motu aquarum per canales quoscunque fluentium," *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **2** (1727), 111–125 (1729). The work "hydrodynamics" was created by D. Bernoulli with the publication of his *Hydrodynamica* in 1738. The Indiana University Library does not possess a copy of this book; exhibited instead is a paper, published a few years earlier, which may serve as a specimen of the methods and results.

No general equations or general principles of fluid motion are to be found here. The *Hydrodynamica* is famous for "Bernoulli's theorem,"

$$p + \tfrac{1}{2}\rho v^2 = \text{const.},$$

which is taught in every freshman physics course. In this simple form, however, it is not to be found in the book, nor is its use given any prominence. In both the journal article (p. 112) and the book the analysis rests on the "principle of the conservation of living forces," which Daniel Bernoulli attributes to his father, John Bernoulli. At this time there was a vigorous general controversy regarding *vis viva* (kinetic energy). In the book, the conservation of energy is asserted in the form of the "principle of equality of potential ascent and actual descent." The terms are defined as follows:*

"We shall note that the *potential ascent* of a system whose several particles are moved with an arbitrary velocity signifies the vertical altitude which the

* *Hydrodynamica*, Sectio tertia, §1.

center of gravity of that system would reach if the several particles are conceived to turn their velocities upward and rise as far as they can; that the *actual descent* denotes the vertical altitude through which the center of gravity descends after the several particles had come to rest." The book was praised by Lagrange as "a work which shines with analysis as elegant in its ordering as simple in its results."

The "Bernoulli equation" in its now usual simplicity was first written down by Euler (see §4, 5), who derived it as an integral of his general hydrodynamic equations. In so doing, he generalized it in three ways, obtaining the forms valid in unsteady potential flow, in barotropic potential flow, and (with the constant generally assuming a value differing from one stream-line to another) for steady rotational flow.

That Daniel Bernoulli did not attain the simplicity of a modern engineering treatment, while Euler, beginning with a much more elaborate mathematical structure, by purely analytic manipulations attained the result in its now customary form, should not astonish us. Daniel Bernoulli wrote not only before Euler had made it customary to express general physical principles as mathematical equations, but also before the concept of internal fluid pressure had been created. His hydrodynamical works consist of long sequences of special problems ingeniously solved by means of the energy principle. Many of these problems are of the type which can be treated somewhat more simply nowadays by use of what is now called "Bernoulli's theorem." His father, the old John Bernoulli, in envy at his son's success* immediately wrote his *Hydraulica*, pre-dating it by seven years so as to make his son appear a plagiarizer. Nevertheless, the new work was enthusiastically praised by Euler as the first to use the "true and genuine method" and to be based on "the most certain principles of mechanics." By the former Euler meant the method of infinitesimal elements; by the latter, the momentum principle. In John Bernoulli's treatise we find also the first rather rough use of internal pressure, and the first actual equation equivalent to a fairly general case of the modern "Bernoulli equation." All this work concerned flow in tubes, and in Euler's early hydraulic papers the greater generality of the problems treated led to forms of the result gradually approaching the modern one.

The paper of Daniel Bernoulli which we were considering ends with a discussion of the validity of the two principles by whose application the foregoing special problems have been solved. Bernoulli states that the "conservation of living forces" remains valid as long as there is no friction or other exterior resistance, but that the "proportionality of the velocity to the inverse of the amplitude," while only an approximation, is valid for most vessels, provided there is no motion suddenly created or destroyed. The latter statement is the principle of incompressibility, expressed in an approximate form correct only for flow of incompressible substances in nearly straight pipes.

* Cf. *e.g.* pp. 95–96 of O. Spiess, *Leonhard Euler*, Frauenfeld and Leipzig, 1929.

3. J. L. d'Alembert, *Essai d'une Nouvelle Theorie de la Résistance des Fluides*, Paris, 1752. Of this rare book only two copies, located at the Navy Department Library and at the Brown University Library, are known to be in the United States. The photostat on exhibit is loaned by the Naval Research Laboratory.

The circumstances of composition of this work are mentioned in the introductory section above. It contains the first publication of the equations governing irrotational flow of incompressible fluids in two dimensions. For axially-symmetric flow, these equations are stated in §45, the notations having been explained in §43. The equations for plane motion are given summarily in §73. The ideas and form of presentation are strongly influenced by the work of d'Alembert's lifelong rival, Clairaut, in his *Théorie de la Figure de la Terre, tirée des Principes de l'Hydrostatique*, Paris, 1743, where the principles of hydrostatics are similarly expressed. Like Clairaut, d'Alembert makes use of the principle of equivalent canals.

The *Essai*, despite the fact that it is so elaborate and so burdened with philosophy that very likely it has never been read in its entirety since 1755, is nevertheless a turning point both in physics and in mathematics. It contains the first field description of media in motion, a description which in other hands led quickly to partial differential equations and in particular to potential theory. It contains also what is generally described as d'Alembert's solution of the equation $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$ in the form $\phi=f(x+iy)+g(x-iy)$, although in fact d'Alembert himself did not obtain the partial differential equation, deriving an equivalent result by manipulating differentials. The analysis is contained in §§57–60, which are famous for the first appearance of the "Cauchy-Riemann equations." The reasoning is essentially as follows: if

$$\frac{\partial p}{\partial z}=-\frac{\partial q}{\partial x}, \qquad \frac{\partial p}{\partial x}=\frac{\partial q}{\partial z},$$

then $qdx+pdz$ and $pdx-qdz$ are complete differentials. By purely formal steps it is shown that $p+iq=f(x-iz)$, $p-iq=g(x+iz)$.

In §70 appears the famous "d'Alembert paradox." Put in modern terms, this "paradox" asserts that a submerged smooth body moving at uniform speed in a continuous irrotational flow of an ideal fluid at rest at infinity experiences no resultant force. It had been stated and proved more simply and more generally by Euler in 1745*, and Oseen† has traced it back to Spinoza. All the early statements refer to symmetrical bodies and employ some form of reversibility argument, whereby the pressure on the back side of a body is shown to cancel precisely the pressure at a corresponding point of the front side. However, no such special reasoning is required, and the result holds for smooth bodies of any form. It is essential, however, that the velocity field fall off

* *Neue Grundsätze der Artillerie, aus dem Englischen des Herrn Benjamin Robins übersetzt und mit vielen Anmerkungen versehen*, Berlin, 1745, *Opera omnia* (2) **14**. See Ch. 2, Law 1, remark 3.

† *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Leipzig, 1927. See introd.

sufficiently rapidly at infinity. For incompressible fluids, the necessary order condition is available from potential theory; for subsonic flow of compressible fluids it has been established rigorously only in recent months.

Since he is writing about a new theory of the *resistance* of fluids, and that theory turns out to give no resistance at all in the most natural situations, d'Alembert appears to be rather dismayed. He does not make a great point of the matter, however, and passes on to treat of many other topics at great length but with few concrete results. Twelve years later he published the paradox again* in rather more sensational terms.

In the *Essai*, most of d'Alembert's dark suspicions regarding the theory are limited to the impossibility, rather than the uselessness, of the calculations. But time has shown that ideal hydrodynamics is far from useless. Even in the problem of resistance, d'Alembert turned out to be wrong. While he claimed to have shown that all motions are irrotational, three years later Euler proved (§5) that irrotational motion is only a very special case; and in rotational motion, the d'Alembert paradox no longer holds. Through proper development and exploitation of this idea, the modern theory of airplane lift has developed.

4. L. Euler, "Principia motus fluidorum" (1752–1755), *Acta Academiae Imperialis Scientiarum Petropolitensis*, 6 (1756–1757), 271–311 (1761). This paper contains Euler's first published treatment of the general problem of fluid motion. In 1752, shortly after the Berlin Academy's rejection of all pieces contesting for the prize on the resistance of fluids, Euler read before it a paper *De motu fluidorum in genere*, and Eneström regarded this fact, which was ascertained by Jacobi, as sufficient evidence to date the present paper 1752. Another paper of Euler, itself dated 1755 (see §5), refers to this paper by its actual title.

The first part of the paper calculates in detail the change in volume of an infinitesimal tetrahedron carried by the motion through an infinitesimal time. Setting this change equal to zero yields a general equation expressing incompressibility. The calculation is carried out first in two dimensions, then in three. On page 286 is the conclusion of the three dimensional treatment, with the resulting *equation of continuity* in a notation still used at the present day:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

where u, v, w are the velocity components.

The second part of the paper introduces rather hastily the concept of fluid pressure and derives the dynamical equations for ideal fluids.

The work is dense with ideas waiting to be worked out. Among its contents we may observe the first occurrence of the secular determinant and its expansion, the introduction of the velocity potential and the acceleration potential,

* Suite des recherches sur le mouvement des fluides, *Opuscules mathématiques*, 5 (1768), no. 34, 132–170. See §I.

derivation of "Laplace's equation" for the former (cf. §3), construction of the general polynomial harmonic of degree n, derivation of "Helmholtz's vorticity equation" as a condition of integrability for the pressure field, derivation of "Killing's equation" for rigid motion, derivation of "Bernoulli's equation" for unsteady irrotational motions, a theory of fluid friction, derivation of the theory of flow in narrow tubes. It is interesting to notice that in this paper Euler concurs with d'Alembert's decision that all fluid motions possess a velocity-potential (irrotational flow).

5. L. Euler, "Principes généraux du mouvement des fluides," *Mémoires de l'Académie Royale des Sciences*, Berlin, **11** (1755), 274–315 (1757). This paper is the second of three parts of a treatise on the mechanics of fluids, published in consecutive pages in the same volume of the journal. In the first part Euler established the partial differential equations of hydrostatics and their general solution. In order to do so, he introduced the concept of *pressure* as a field acting upon an imaginary closed boundary and mechanically equivalent to the action of the fluid exterior to the boundary upon that interior to it. Previous authors had regarded the pressure as the weight of a unit column of fluid, or the height to which water would rise if the vessel were punctured. The distinction is extremely important, and it is Euler's concept of pressure which is used in modern hydrodynamics. Neither Clairaut's hydrostatics (1743) nor d'Alembert's hydrodynamics used the pressure at all, except possibly by implication. Their general principles consist in statements that certain differentials shall be exact. Euler was the first to conceive any branch of mathematical physics as a field theory governed by partial differential equations. Lagrange* in comparing the work of d'Alembert and Euler wrote:

"But these equations did not yet possess all the generality and the simplicity of which they were susceptible. It is to Euler that we owe the first general formulae for the motion of fluids, founded on the laws of their equilibrium, and presented with the simple and luminous notation of partial differences. By this discovery, all the mechanics of fluids was reduced to a single point of analysis. . . ." The importance of the pressure, even for hydrostatics alone, was put so highly by Todhunter† that he wrote:

"Before Euler thus illustrated [*i.e.*, illuminated] the subject, there had been *demonstrations* in hydrostatics, but I cannot consider that these demonstrations were altogether intelligible."

The paper shown here clarifies and generalizes the pioneer work of 1752. Unlike the latter, it is smooth and pleasant reading, expressed in simple and easy notations. Both physically and mathematically it shows on every page what Fourier described as "that admirable clarity which is the principal character of all Euler's writings." The hydrodynamical equations are derived from what Euler called "the first axioms of mechanics," *viz.*, the indestructibility of

* *Loc. cit.*, but only in the second (1815) and later editions.

† *Op. cit.*, §301.

matter and momentum.

On p. 286 Euler obtains his general dynamical equations for ideal fluids:

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \text{ etc.}$$

Previously he had derived the continuity equation for the compressible case:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

and he emphasizes the generality and the completeness of the set of four equations, complemented by a relation between pressure and density.

After a great many interesting tentatives and discoveries, Euler at the end of the paper concludes that mathematical analysis must be greatly improved before further progress in fluid dynamics can be made. "We see . . . how far distant we still are from a complete knowledge of the motion of fluids, and what I have just explained is only a feeble beginning. Nevertheless all that the theory of fluids includes is contained in the two equations presented above, so that it is not the principles of mechanics which we need for the pursuit of these researches, but analysis alone, which is not yet sufficiently cultivated for this end. One sees clearly what discoveries remain for us to make in this science before we can arrive at a more perfect theory of the motion of fluids."

6. L. Euler, "Recherches sur la propagation des ébranlemens dans une milieu élastique," *Mélanges de Turin*, 2^2 (1760–1761), 1–10 (1762); *Opera* (2) **10**, 255–263; *Oeuvres de Lagrange* **14**, 178–188. Throughout the latter part of his life Euler several times returned to the theory of fluids, either to extend its applications or to strengthen its foundations. His final work on the subject is a definitive Latin treatise in four lengthy instalments, published partly after his death.

The incorrect value for the speed of sound obtained by Newton in the *Principia* gave rise to a century and a half of controversy. It occurred to Euler that acoustics should be a branch of hydrodynamics, and that a proper value for the speed of sound should be obtainable from the hydrodynamical field equations. The problem had been taken up meanwhile by Lagrange, then an unknown young man at the beginning of his mathematical career. Lagrange had written to Euler on this subject as well as on the calculus of variations, and Euler replied with his usual generosity and candor.

This paper is the 1762 printing of one of Euler's letters to Lagrange, dated January 1, 1760. In it he derives a new set of hydrodynamical equations, expressed in terms of the positions of the fluid particles at some arbitrary initial instant. These initial positions are what are now called the "Lagrangian coordinates." The misnomer arose from the fact that Lagrange in the *Méchanique Analitique* did not quote the specific sources of any of his hydrodynamical

material, permitting the reader to gain the impression that much of it appeared there for the first time, while in fact all of it is contained in earlier papers by Euler or by Lagrange himself.*

Euler's second development of the field equations for fluids is in some ways more complete than the first, since it employs a more detailed kinematical description. All the principal equations, including the general formulae for inversion of partial differentiations, are given in the letter.

As is now easily understood, the proper speed of sound was not and could not be calculated from the hydrodynamical field equations alone. The result of Euler and Lagrange's researches was to establish the wave equation on a solid foundation, at the same time showing that the essence of the problem of the speed of sound lies in the form of the relation between pressure and density.

7. C.-L.-M.-H. Navier, "Mémoire sur les lois du mouvement des fluides" (1822), *Mémoires de l'Académie des Sciences de l'Institut de France*, (2) **6**, 389–440, (1827). Euler's equations when applied to flow in a long straight pipe possess an infinite number of solutions; in fact, any continuously differentiable flow with straight parallel stream-lines is such a solution. However, a physical fluid when forced down such a pipe at sufficiently low speed assumes a single, perfectly definite velocity distribution. It follows that in Euler's equations some essential physical phenomenon has been neglected. In fact, Euler's equations refer to fluids without dissipative mechanisms, sometimes called "perfect" or "ideal" fluids. While such fluids closely approximate the behavior of physical fluids in many circumstances, in cases where internal friction is important a more accurate mathematical model is required. For example, neglect of the adherence of fluids to solid boundaries is one of the reasons for the "d'Alembert paradox" (§3).

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* Actually it is almost certain that Euler *began* his work on the general theory of fluids with the material description, but soon took up the simpler spatial description, which alone is used in his earliest publication. D'Alembert in the last section (§160) of the *Essai* refers sarcastically to "a manuscript theory on the current of rivers," which, he says, could not have fallen into his hands until August, 1750. He says further that "it would not even be impossible that the method presented in my work was unknown to the author of the memoir of which I speak, and did not help him in his researches on the current of rivers." Now there is a paper by Euler called *Recherches sur le mouvement de rivières*, which appears on pp. 101–118 of the Berlin Mémoires for 1760, not published until 1767. This paper presents internal evidence of being written before the *Principia motus* (1752); in particular, it employs a strange set of units used by Euler at this time, but abandoned in later papers; the formal structure is incomplete; and the specific flows considered as examples are trivial. According to Jacobi (quoted by Eneström, *Op. cit.*), a paper of this title was presented to the Berlin Academy on May 6, 1751. That it was *not* published sooner may be explained by the fact that in content it is a fragmentary and unsuccessful attempt, doubtless cast aside by Euler as he progressed in his researches. That it *was* published in 1767 may well be explained by Euler's angry departure from Berlin in 1766, depriving the academy of half its customary volume of MSS for publication.

If we can accept the date 1751 for this paper, we can set that year as the discovery of the material description. For it is this description alone which is employed in this paper.

By the year 1820 the foregoing facts were becoming rather dimly apprehended. In 1821 Navier, a French engineer, constructed imaginary models both for solid bodies and for fluids by regarding them as nearly static assemblages of "molecules," mass-points obeying certain intermolecular force laws. Forces of cohesion were regarded as arising from summation of the multitudinous intermolecular actions. Such models were not new, having occurred in philosophical or qualitative speculations for millennia past. Navier's magnificent achievement was to put these notions into sufficiently concrete form that he could derive equations of motion from them.

On p. 414 occur the basic dynamical equations, now called "the Navier-Stokes equations," which govern the motion of incompressible viscous fluids. In modern vector notation these read:

$$\mathbf{f} - \frac{1}{\rho}\operatorname{grad} p - \frac{\mu}{\rho}\nabla^2\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\mathbf{v},$$

where $\mathbf{v}$ is the velocity vector and μ is the *shear viscosity*, a coefficient for which Navier gave an expression in terms of the intermolecular forces, but which is now usually regarded as an experimental function of temperature for each type of fluid. Navier's equation differs from Euler's in the presence of the term $(\mu/\rho)\nabla^2\mathbf{v}$, representing the effect of the internal friction, and, in so doing, raising the order of the differential equations from 1 to 2.

8. A. Cauchy, "Sur les équations qui expriment les conditions de l'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique," *Exercises de Mathématique*, **3** (1828); *Oeuvres* (2) **8,** 195–226. Fresnel's optical discoveries and the elastic theory of the ether which he proposed concurrently excited the intense interest of the entire scientific world. Immediately Cauchy took up the theoretical side of the subject, generalizing it profoundly and at the same time expressing the principles and results with the clarity and simplicity of the Eulerian tradition. The major results were announced in 1823, but proofs and explanations did not begin to appear until 1827, the second year of Cauchy's remarkable *Exercises*, annual volumes of short original papers by Cauchy alone, where jewels of number theory alternate with mechanics, complex function theory with algebra, pure geometry with the theory of infinite series.

Cauchy's first view regards matter as a continuous medium whose mechanical response is specified through *internal pressures*. These differ from Euler's hydrostatic pressure only in that their local resultant need not be normal to the surface element across which they act. This concept has the simplicity of genius. Its profound originality can be grasped only when one realizes that a whole century of brilliant geometers had treated very special elastic problems in very complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became the foundation of the mechanics of distributed matter.

Cauchy's main interest was in optical application, and he put great efforts into the elastic theory of light. §III of this paper contains a by-product, mentioned briefly in passing. If a continuous body is "altogether devoid of elasticity," the stress tensor t_{ij} assumes the form (Cauchy's eq. (95) (97), written in indicial notation)

$$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$$

where v_i is the velocity vector. When these are put into Cauchy's general form of the momentum principle,

$$t_{ij,j} + f_i = \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right),$$

there result dynamical equations which differ from those for viscous compressible fluids only in lacking the hydrostatic pressure term. Cauchy apparently regarded them as suitable for very soft solid bodies. If $\lambda = 0$, while μ is regarded as a suitable function of stress, rather than of temperature, they become the equations of plasticity proposed by St. Venant (1870), Levy (1870), and v. Mises (1913), but no such interpretation was intended by Cauchy, who stated that for homogeneous bodies μ is a constant.

9. S.-D. Poisson, "Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides" (1829), *Journal de l'Ecole Polytechnique*, **13** (cahier 20), 1–174 (1831). There is no doubt that Poisson first learned of the general equations of elasticity and fluid dynamics from the memoirs of Navier, which were read at the French Academy in 1821 and 1822. However, Poisson contended that despite the correctness of the final results, Navier's analysis was wholly faulty. The lives of both these savants were embittered by the senseless public controversy which followed: Navier repeatedly claimed priority, Poisson repeatedly denied it, on the ground that a result derived by false reasoning is not derived at all. It is interesting to note the change of fashions in physics: Poisson, who was a physicist, based his attack mainly on Navier's having replaced summations over the individual particles by integrations over volumes.

Neither critic nor criticised argued soundly. Both viewed a liquid as a nearly static assembly of Newtonian mass-points; they differed one from the other only in details which now seem unimportant, since such a picture of the liquid state has been entirely abandoned, and both derivations must be rejected. Their models are of interest only as examples of how a wholly wrong molecular theory can lead to wholly right gross conclusions—a fact often overlooked in physical circles today.

Although Poisson's part in this controversy was not admirable, his physical sagacity and analytical skill did not fail to get new results. Just as it was Navier, and Navier alone, who discovered the equations for viscous incompressible

fluids, so it was Poisson, and Poisson alone, who discovered those for compressible viscous fluids. In modern notations, Poisson's stress formula is

$$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}).$$

Here occur both the pressure p, as in Euler's and Navier's equations, and the two viscosities λ and μ, as in Cauchy's equations. While the generalization is quite straightforward from the continuum viewpoint, it must be remembered that Poisson labored with an intricate molecular model, which he endeavored to manipulate both with physical propriety and with mathematical exactness.

10. G. G. Stokes, "On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solides" (1845), *Transactions of the Cambridge Philosophical Society*, **8** (1844–1849), 287–319; *Papers* **1**, 75–129. When Stokes wrote this paper, the general equations for elastic bodies and for viscous fluids had been in use in France for some twenty years, but they were not yet employed in England. It may be said that in England a result is not counted as known until an Englishman has rediscovered it. However, unlike some more recent compatriots, Stokes read, understood, and admired the continental literature, particularly the work of the great French mathematical physicsts, whose methods he brought to a Cambridge stultified by traditional adherence to the faults rather than the virtues of Newton's methods. In this paper as in his other works Stokes did not disdain to add to the results of deep physical insight and high analytical skill the indications of thorough scholarship, and in particular he discussed in detail the work of Cauchy and Poisson on the same subjects.

Stokes's contribution, as far as the general equations are concerned, is an extraordinarily natural and incisive discussion of the concepts. In §2 we find a "Principle" which expresses the notion of internal friction in its most general form:

"That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned." A hundred years later this same principle, expressed mathematically in tensor form, was shown by Reiner and Rivlin* to lead to much more general equations of fluid dynamics and to predict extremely interesting new physical phenomena. Stokes himself, content with a linear approximation, derived the

* M. Reiner, A mathematical theory of dilatancy, Am. J. Math. **67**, 350–362 (1945).

R. S. Rivlin, The hydrodynamics of non-Newtonian fluids, Proc. R. Soc. London (A) **193**, 260–281 (1948), Proc. Cambr. Phil. Soc. 45, 88–91, (1949).

See also C. Truesdell, A new definition of a fluid, J. Math. Pures Appl. (9) **29**, 215–244, (1950); **30**, 111–158 (1951).

same equations as had Poisson, but in so doing he put the theory on a sound and clear phenomenological basis. As far as the received theory of fluids with linear viscous response is concerned, this paper was final.

A part of Stokes's discussion concerns the "Stokes relation" $3\lambda+2\mu=0$ between the two viscosities. While put forward with reserve by Stokes, and later virtually retracted by him, unfortunately this specialization was universally accepted in the physical literature until very recently. Experimental results, however, show that it can be correct only for monatomic gases at most.

The remainder of the paper contains: a determination of the boundary condition for viscous fluids; solution of the problem of decay of free plane infinitesimal waves in viscous fluids, whence arose later the theory of ultrasonic absorption and dispersion; correction of Newton's erroneous solution of the problem of a cylinder rotating in a fluid (see §1); solution of the problem of flow in a straight pipe; a scholarly and penetrating discussion of the theorem of Lagrange in ideal hydrodynamics; development of elasticity theory; and a critique of the work of Poisson.

Postscript. It would certainly be wrong to conclude that analysis of the principles fundamental to the hydrodynamical equations stopped in 1845. True, such is the impression which one gains often enough from current textbooks, some of which produce the Navier-Stokes equations as empirical results established by experiment. As we have seen, while *experience* is indeed the basis of hydrodynamics, *experiment* played no direct part in the development of the general equations. The mathematical analysis of Navier, Cauchy, Poisson, and Stokes makes it plain that their results are only *first approximations* to those which would follow from a more general theory. It was natural enough that the main effort in hydrodynamics since 1845 should be directed toward obtaining solutions of the equations of Euler or Navier-Stokes and in comparing these solutions with experimentally measured values. As a result of this vigorous and thorough exploitation, we now have a fair idea both of the mathematics of classical hydrodynamics and of the range of its physical validity. But from time to time there have been attempts at precision of the more general ideas standing in the shadows behind Stokes's Principle. During the past seven years this question, whose theory might be named *Non-linear viscosity*, has become an active and vigorous field of modern mechanics.*

* A skeleton history is given in §18 of my paper cited above. A treatise on the state of the theory up to January, 1952, is given in Chapter V of my paper, The mechanical foundations of elasticity and fluid dynamics, *J. Rational Mech. Anal.* **1**, 125–300 (1952); **2**, 593–616 (1953).

SOME NEW MULTIPLY PERFECT NUMBERS

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Using a modified form of a method employed by Poulet [1, pp. 10–12], the authors have obtained the sixty-three new multiply perfect numbers listed below.† The main modification to Poulet's method consists of tabulating, for each prime $p<100$, some of the powers q^a of primes q such that the sum of the divisors of q^a leads, by Poulet's considerations [1, p. 12] to the factor p. This additional table was of considerable help to us in our findings.

The list given here supplements that of Poulet's [1, pp. 14–25], which includes all lists previously published. From each number marked with an asterisk, another is obtained by substituting $19^4 \cdot 151 \cdot 911$ for $19^2 \cdot 127$.

Class 4 (Sum of divisors five times the number)

D		
	1	$2^{41} \cdot 3^{12} \cdot 7^4 \cdot 11^2 \cdot 13^4 \cdot 17^2 \cdot 19^4 \cdot 43 \cdot 103 \cdot 127 \cdot 151 \cdot 191 \cdot 271 \cdot 307 \cdot 337 \cdot 467 \cdot 617 \cdot 911 \cdot 2801 \cdot 5419 \cdot 30941 \cdot 398581 \cdot 797161$
	2 and 3	$*2^{42} \cdot 3^{14} \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 19^2 \cdot 31 \cdot 41 \cdot 43 \cdot 61^3 \cdot 83 \cdot 127 \cdot 163 \cdot 331 \cdot 431 \cdot 1861 \cdot 2281 \cdot 4561 \cdot 9719 \cdot 2099863$
	4 and 5	$*2^{51} \cdot 3^{18} \cdot 5^4 \cdot 11 \cdot 17 \cdot 19^2 \cdot 47 \cdot 53 \cdot 71 \cdot 79 \cdot 127 \cdot 157 \cdot 269 \cdot 607 \cdot 683 \cdot 1213 \cdot 1597 \cdot 1613 \cdot 2731 \cdot 8191 \cdot 36389 \cdot 363889$

Class 5 (Sum of divisors six times the number)

E		
	1 and 2	$*2^{19} \cdot 3^5 \cdot 5^2 \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19^2 \cdot 31^2 \cdot 41 \cdot 43 \cdot 83 \cdot 127 \cdot 331$
	3	$2^{19} \cdot 3^6 \cdot 5^5 \cdot 7^7 \cdot 11^2 \cdot 13 \cdot 19 \cdot 31^2 \cdot 41 \cdot 43 \cdot 83 \cdot 331 \cdot 601 \cdot 1201$
	4	$2^{20} \cdot 3^2 \cdot 5^2 \cdot 7^5 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 31 \cdot 43 \cdot 127^2 \cdot 271 \cdot 307 \cdot 337 \cdot 5419$
	5	$2^{20} \cdot 3^2 \cdot 5^3 \cdot 7^5 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 43 \cdot 127^2 \cdot 271 \cdot 307 \cdot 337 \cdot 5419$
	6	$2^{23} \cdot 3^9 \cdot 5^3 \cdot 7^6 \cdot 11^7 \cdot 13^3 \cdot 17^2 \cdot 29 \cdot 61^2 \cdot 97 \cdot 131 \cdot 241 \cdot 263 \cdot 307 \cdot 523 \cdot 4733 \cdot 7321$
	7	$2^{27} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19^4 \cdot 29 \cdot 31 \cdot 43 \cdot 113 \cdot 127 \cdot 151 \cdot 911$
	8	$2^{27} \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 19^4 \cdot 29 \cdot 43 \cdot 71 \cdot 113 \cdot 127 \cdot 151 \cdot 911$
	9	$2^{27} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19^4 \cdot 29 \cdot 41 \cdot 43 \cdot 113 \cdot 127 \cdot 151 \cdot 911$
	10	$2^{29} \cdot 3^{13} \cdot 5^2 \cdot 7^5 \cdot 11^4 \cdot 17 \cdot 19^2 \cdot 31^2 \cdot 41 \cdot 43 \cdot 127 \cdot 151 \cdot 163 \cdot 179 \cdot 307 \cdot 331^2 \cdot 547^2 \cdot 613 \cdot 1093 \cdot 2617 \cdot 3221 \cdot 5233$
	11	$2^{29} \cdot 3^{13} \cdot 5^5 \cdot 7^7 \cdot 11^4 \cdot 17 \cdot 19 \cdot 31^2 \cdot 41 \cdot 43 \cdot 151 \cdot 163 \cdot 179 \cdot 307 \cdot 331^2 \cdot 547^2 \cdot 601 \cdot 613 \cdot 1093 \cdot 1201 \cdot 2617 \cdot 3221 \cdot 5233$
	12	$2^{31} \cdot 3^{15} \cdot 5^5 \cdot 7^6 \cdot 11^4 \cdot 17^2 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 83 \cdot 97 \cdot 179 \cdot 193 \cdot 257 \cdot 263 \cdot 307 \cdot 331 \cdot 3221 \cdot 4733 \cdot 65537$

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† After this paper was sent to the Editor, the authors learned that four of the numbers listed here had been obtained (but never published) by the late Paul Poulet and that Alan L. Brown has independently discovered, among others, twenty-one of the numbers in our list. The reader is referred for further details to an article by Brown on "Multiply Perfect Numbers" which is to appear soon in *Scripta Mathematica*.

13	$2^{32}\cdot 3^{12}\cdot 5^{7}\cdot 7^{4}\cdot 13^{3}\cdot 17^{2}\cdot 19\cdot 23\cdot 29\cdot 79\cdot 89\cdot 103\cdot 157\cdot 313\cdot 467\cdot 617$ $\cdot 2141\cdot 2801\cdot 398581\cdot 599479\cdot 797161$
14	$2^{36}\cdot 3^{13}\cdot 5^{2}\cdot 7^{9}\cdot 11^{2}\cdot 13^{4}\cdot 19\cdot 31^{2}\cdot 41\cdot 83\cdot 163\cdot 191^{2}\cdot 223\cdot 307\cdot 331$ $\cdot 467\cdot 547^{2}\cdot 613\cdot 1093\cdot 2801\cdot 7019\cdot 30941\cdot 112303\cdot 898423$ $\cdot 616318177$
15	$2^{36}\cdot 3^{15}\cdot 5^{2}\cdot 7^{9}\cdot 11\cdot 13^{4}\cdot 17\cdot 31^{2}\cdot 41\cdot 83\cdot 97\cdot 191^{2}\cdot 193\cdot 223\cdot 331\cdot 467$ $\cdot 2801\cdot 7019\cdot 30941\cdot 112303\cdot 898423\cdot 616318177$
16	$2^{41}\cdot 3^{7}\cdot 5^{6}\cdot 7^{3}\cdot 13^{2}\cdot 17\cdot 19\cdot 31\cdot 41\cdot 43^{2}\cdot 61\cdot 79\cdot 127\cdot 257\cdot 271\cdot 337$ $\cdot 631\cdot 5419\cdot 19531$
17	$5D_1$
18 and 19	$5D_2$ and $5D_3$
20	$2^{44}\cdot 3^{6}\cdot 5^{2}\cdot 7\cdot 11^{3}\cdot 19^{2}\cdot 23\cdot 31^{4}\cdot 37\cdot 47\cdot 61\cdot 73\cdot 79\cdot 127\cdot 137\cdot 151$ $\cdot 241\cdot 547\cdot 631\cdot 1093\cdot 17351\cdot 23311$
21	$2^{44}\cdot 3^{10}\cdot 5^{2}\cdot 7\cdot 11^{3}\cdot 19^{2}\cdot 23\cdot 31^{4}\cdot 37\cdot 47\cdot 61\cdot 73\cdot 79\cdot 107\cdot 127\cdot 151$ $\cdot 241\cdot 631\cdot 3851\cdot 17351\cdot 23311$
22 and 23	$*2^{50}\cdot 3^{2}\cdot 5^{3}\cdot 7^{5}\cdot 11^{2}\cdot 13^{3}\cdot 17^{2}\cdot 19^{2}\cdot 43\cdot 67\cdot 103\cdot 127\cdot 139\cdot 307\cdot 2143$ $\cdot 11119\cdot 131071$
24	$2^{50}\cdot 3^{3}\cdot 5^{6}\cdot 7^{3}\cdot 11\cdot 13\cdot 17\cdot 19\cdot 43\cdot 67\cdot 103\cdot 139\cdot 257\cdot 2143\cdot 11119$ $\cdot 19531\cdot 131071$
25 and 26	$*2^{51}\cdot 3^{20}\cdot 5^{7}\cdot 7^{2}\cdot 13^{2}\cdot 19^{2}\cdot 23^{2}\cdot 31\cdot 41\cdot 53\cdot 61\cdot 79\cdot 127\cdot 137\cdot 157^{2}$ $\cdot 269\cdot 313\cdot 409\cdot 547\cdot 683\cdot 827\cdot 1093\cdot 1613\cdot 2731\cdot 8191\cdot 8269$ $\cdot 36809\cdot 368089$
27 and 28	$*2^{52}\cdot 3^{7}\cdot 5^{6}\cdot 7^{3}\cdot 11^{3}\cdot 19^{2}\cdot 31\cdot 37\cdot 41\cdot 43^{2}\cdot 61\cdot 79\cdot 127\cdot 139\cdot 257\cdot 263$ $\cdot 277\cdot 439\cdot 631\cdot 877\cdot 1753\cdot 3181\cdot 6361\cdot 19531\cdot 69431\cdot 20394401$
29	$2^{56}\cdot 3^{16}\cdot 5^{5}\cdot 7^{7}\cdot 11\cdot 13^{2}\cdot 17\cdot 31^{2}\cdot 43\cdot 61\cdot 83\cdot 331\cdot 601\cdot 719\cdot 1201$ $\cdot 1619\cdot 1871\cdot 16189\cdot 32377\cdot 34511\cdot 524287\cdot 1212847$

Class 6 (Sum of divisors seven times the number)

F	1	$2^{31}\cdot 3^{20}\cdot 5^{8}\cdot 7^{4}\cdot 11^{4}\cdot 13^{2}\cdot 17^{2}\cdot 19\cdot 23\cdot 31^{2}\cdot 41\cdot 43\cdot 61\cdot 83\cdot 137\cdot 179$ $\cdot 257\cdot 307\cdot 331^{2}\cdot 409\cdot 467\cdot 547\cdot 829\cdot 1093\cdot 2617\cdot 2801\cdot 3221$ $\cdot 5233\cdot 36809\cdot 65537\cdot 368089$
	2	$2^{32}\cdot 3^{17}\cdot 5^{5}\cdot 7^{6}\cdot 11\cdot 13^{2}\cdot 17\cdot 19^{3}\cdot 23\cdot 29\cdot 31^{2}\cdot 37\cdot 61\cdot 83\cdot 89\cdot 181\cdot 263$ $\cdot 331\cdot 379\cdot 757\cdot 2141\cdot 4733\cdot 599479$
3 and	4	$*2^{35}\cdot 3^{15}\cdot 5^{6}\cdot 7^{6}\cdot 11^{4}\cdot 13\cdot 17^{2}\cdot 19^{2}\cdot 29\cdot 37^{2}\cdot 41\cdot 43\cdot 67\cdot 73\cdot 97\cdot 109$ $\cdot 127\cdot 179\cdot 193\cdot 257\cdot 263\cdot 307\cdot 3221\cdot 4733\cdot 19531$
	5	$2^{35}\cdot 3^{20}\cdot 5^{6}\cdot 7^{4}\cdot 11^{2}\cdot 13^{4}\cdot 17\cdot 19^{2}\cdot 23\cdot 37^{2}\cdot 41\cdot 43\cdot 67\cdot 73\cdot 109\cdot 137$ $\cdot 181\cdot 191\cdot 257\cdot 409\cdot 467\cdot 547\cdot 1093\cdot 2801\cdot 19531\cdot 30941$ $\cdot 36809\cdot 368089$
	6	$2^{39}\cdot 3^{9}\cdot 5^{6}\cdot 7^{5}\cdot 11^{3}\cdot 13^{3}\cdot 17^{3}\cdot 19^{3}\cdot 29^{2}\cdot 31\cdot 41\cdot 43^{2}\cdot 61^{2}\cdot 67\cdot 79\cdot 97$ $\cdot 181\cdot 257\cdot 631\cdot 2203\cdot 19531\cdot 30841\cdot 61681$
	7	$2^{39}\cdot 3^{12}\cdot 5^{4}\cdot 7^{8}\cdot 11^{3}\cdot 13\cdot 17^{2}\cdot 19^{5}\cdot 29\cdot 31^{2}\cdot 37\cdot 41\cdot 61\cdot 71\cdot 83\cdot 103$ $\cdot 127\cdot 307\cdot 331\cdot 617\cdot 1063\cdot 2203\cdot 30841\cdot 61681\cdot 398581\cdot 797161$

8	$2^{39}\cdot 3^{12}\cdot 5^{4}\cdot 7^{8}\cdot 11^{3}\cdot 13^{2}\cdot 17^{2}\cdot 19^{5}\cdot 29\cdot 31\cdot 37\cdot 41\cdot 61^{2}\cdot 71\cdot 97\cdot 103$ $\cdot 127\cdot 307\cdot 617\cdot 1063\cdot 2203\cdot 30841\cdot 61681\cdot 398581\cdot 797161$
9	$2^{40}\cdot 3^{20}\cdot 5^{6}\cdot 7^{4}\cdot 11\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 31^{2}\cdot 37\cdot 41\cdot 43\cdot 83\cdot 103\cdot 137$ $\cdot 257\cdot 331\cdot 409\cdot 467\cdot 547\cdot 557\cdot 1093\cdot 2801\cdot 7621\cdot 13367\cdot 15241$ $\cdot 36809\cdot 368089\cdot 164511353$
10	$2^{43}\cdot 3^{17}\cdot 5^{6}\cdot 7^{2}\cdot 11\cdot 13^{2}\cdot 17\cdot 19^{7}\cdot 23\cdot 31\cdot 37\cdot 43\cdot 61\cdot 71\cdot 89\cdot 151\cdot 181$ $\cdot 199\cdot 257\cdot 379\cdot 397\cdot 683\cdot 757\cdot 2113\cdot 3833\cdot 19531$
11	$2^{43}\cdot 3^{20}\cdot 5^{9}\cdot 7^{5}\cdot 11\cdot 13^{2}\cdot 19^{3}\cdot 23^{2}\cdot 29\cdot 31\cdot 41\cdot 43^{2}\cdot 61\cdot 71\cdot 79^{2}\cdot 89\cdot 137$ $\cdot 151\cdot 181\cdot 199\cdot 397\cdot 409\cdot 521\cdot 547\cdot 631\cdot 683\cdot 1093\cdot 2113$ $\cdot 36809\cdot 368089$
12	$2^{44}\cdot 3^{8}\cdot 5^{5}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17^{2}\cdot 19^{3}\cdot 29\cdot 31^{3}\cdot 37^{2}\cdot 47\cdot 67\cdot 73\cdot 79\cdot 151$ $\cdot 181\cdot 263\cdot 307\cdot 379\cdot 631\cdot 757\cdot 4733\cdot 23311$
13	$2^{45}\cdot 3^{20}\cdot 5^{4}\cdot 7^{3}\cdot 11^{3}\cdot 13^{2}\cdot 17\cdot 19^{2}\cdot 23\cdot 41\cdot 43\cdot 47\cdot 61^{2}\cdot 71\cdot 97\cdot 127$ $\cdot 137\cdot 151\cdot 197\cdot 271\cdot 409\cdot 547\cdot 1093\cdot 5419\cdot 36809\cdot 178481$ $\cdot 368089\cdot 2796203$
14	$2^{46}\cdot 3^{18}\cdot 5^{5}\cdot 7^{5}\cdot 11\cdot 13\cdot 17\cdot 19^{3}\cdot 23\cdot 31^{2}\cdot 37\cdot 43\cdot 47\cdot 61\cdot 83\cdot 89\cdot 181$ $\cdot 331\cdot 607\cdot 1213\cdot 1597\cdot 2351\cdot 4513\cdot 36389\cdot 363889\cdot 442151$ $\cdot 13264529$
15	$2^{46}\cdot 3^{18}\cdot 5^{5}\cdot 7^{5}\cdot 11\cdot 13^{2}\cdot 17\cdot 19^{3}\cdot 23\cdot 31\cdot 37\cdot 43\cdot 47\cdot 61^{2}\cdot 89\cdot 97\cdot 181$ $\cdot 607\cdot 1213\cdot 1597\cdot 2351\cdot 4513\cdot 36389\cdot 363889\cdot 442151$ $\cdot 13264529$
16	$2^{46}\cdot 3^{18}\cdot 5^{5}\cdot 7^{8}\cdot 11\cdot 13\cdot 17^{2}\cdot 19^{3}\cdot 23\cdot 31^{2}\cdot 37^{2}\cdot 47\cdot 61\cdot 67\cdot 83\cdot 89\cdot 181$ $\cdot 307\cdot 331\cdot 607\cdot 1063\cdot 1213\cdot 1597\cdot 2351\cdot 4513\cdot 36389\cdot 363889$ $\cdot 442151\cdot 13264529$
17	$2^{46}\cdot 3^{18}\cdot 5^{5}\cdot 7^{8}\cdot 11\cdot 13^{2}\cdot 17^{2}\cdot 19^{3}\cdot 23\cdot 31\cdot 37^{2}\cdot 47\cdot 61^{2}\cdot 67\cdot 89\cdot 97$ $\cdot 181\cdot 307\cdot 607\cdot 1063\cdot 1213\cdot 1597\cdot 2351\cdot 4513\cdot 36389\cdot 363889$ $\cdot 442151\cdot 13264529$
18 and 19	$*2^{46}\cdot 3^{20}\cdot 5^{6}\cdot 7^{6}\cdot 11\cdot 13\cdot 19^{2}\cdot 23^{2}\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43^{2}\cdot 61\cdot 79^{2}\cdot 89\cdot 127$ $\cdot 137\cdot 257\cdot 263\cdot 409\cdot 547\cdot 631\cdot 1093\cdot 2351\cdot 4513\cdot 4733\cdot 19531$ $\cdot 36809\cdot 368089\cdot 442151\cdot 13264529$
20	$2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{7}\cdot 11\cdot 13\cdot 19\cdot 23^{2}\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43^{2}\cdot 61\cdot 71\cdot 79^{2}\cdot 89$ $\cdot 137\cdot 409\cdot 521\cdot 547\cdot 601\cdot 631\cdot 1093\cdot 1201\cdot 2351\cdot 4513\cdot 36809$ $\cdot 368089\cdot 442151\cdot 13264529$
21 and 22	$*2^{62}\cdot 3^{14}\cdot 5^{2}\cdot 7^{8}\cdot 11^{7}\cdot 13\cdot 17\cdot 19^{2}\cdot 31^{2}\cdot 37^{2}\cdot 41\cdot 43\cdot 61\cdot 67\cdot 83\cdot 127$ $\cdot 131\cdot 139\cdot 163\cdot 263\cdot 277\cdot 331\cdot 439\cdot 523\cdot 877\cdot 1063\cdot 1753$ $\cdot 2281\cdot 3181\cdot 4561\cdot 6361\cdot 7321\cdot 69431\cdot 20394401$
23 and 24	$*2^{62}\cdot 3^{14}\cdot 5^{2}\cdot 7^{8}\cdot 11^{7}\cdot 13^{2}\cdot 17\cdot 19^{2}\cdot 31\cdot 37^{2}\cdot 41\cdot 43\cdot 61^{2}\cdot 67\cdot 97\cdot 127$ $\cdot 131\cdot 139\cdot 163\cdot 263\cdot 277\cdot 439\cdot 523\cdot 877\cdot 1063\cdot 1753\cdot 2281$ $\cdot 3181\cdot 4561\cdot 6361\cdot 7321\cdot 69431\cdot 20394401$
25	$2^{63}\cdot 3^{15}\cdot 5^{7}\cdot 7^{4}\cdot 11^{4}\cdot 13^{2}\cdot 17\cdot 19\cdot 31\cdot 37\cdot 41\cdot 43\cdot 61\cdot 79\cdot 97\cdot 139\cdot 157$ $\cdot 179\cdot 193\cdot 263\cdot 277\cdot 313\cdot 439\cdot 467\cdot 877\cdot 1753\cdot 2801\cdot 3181\cdot 3221$ $\cdot 6361\cdot 69431\cdot 20394401$

26	$2^{52}\cdot 3^{21}\cdot 5^{5}\cdot 7^{4}\cdot 11^{4}\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 37\cdot 43\cdot 67\cdot 83\cdot 107\cdot 139\cdot 179$ $\cdot 263\cdot 277\cdot 331\cdot 439\cdot 467\cdot 661\cdot 877\cdot 1753\cdot 2801\cdot 3181\cdot 3221$ $\cdot 3851\cdot 6361\cdot 69431\cdot 20394401$
27 and 28	$^{*}2^{63}\cdot 3^{22}\cdot 5^{13}\cdot 7^{6}\cdot 11^{4}\cdot 13\cdot 17^{2}\cdot 19^{2}\cdot 29\cdot 31\cdot 43\cdot 47\cdot 61\cdot 83\cdot 107\cdot 127$ $\cdot 139\cdot 179\cdot 257^{2}\cdot 307\cdot 331\cdot 349\cdot 449\cdot 641\cdot 1087\cdot 3221\cdot 3613$ $\cdot 19531\cdot 27919\cdot 55837\cdot 65537\cdot 335021\cdot 2384579\cdot 3350209$ $\cdot 6700417\cdot 1001523179$
29	$2^{67}\cdot 3^{20}\cdot 5^{7}\cdot 7^{7}\cdot 11\cdot 13^{3}\cdot 17\cdot 31\cdot 37^{2}\cdot 41\cdot 43^{2}\cdot 47\cdot 53\cdot 61\cdot 67\cdot 73\cdot 79^{2}$ $\cdot 83\cdot 137^{2}\cdot 157\cdot 313\cdot 331\cdot 409\cdot 547\cdot 601\cdot 631\cdot 953\cdot 1093\cdot 1201$ $\cdot 13159\cdot 26317\cdot 36809\cdot 43691\cdot 131071\cdot 368089$

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ON TWO THEOREMS OF PLANE TOPOLOGY

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1. The following theorem is stated in [2], p. 55:

THEOREM A. *If C is a continuum in the plane and G is a region whose complement is contained in the outer region determined by C then there is a simply connected subregion of G which contains C.*

(For definitions of the terms used see section 2.) This theorem is of use in applying the Cauchy Integral Theorem ([2], p. 48), since if G is the domain of regularity of a complex valued function and C is a rectifiable closed curve, it follows that if the complement of G is contained in the outer region determined by C then the integral of the function around the curve is zero. Theorem A takes the place, in [2], of some slightly fuzzy remarks in [1], §17, paragraphs 1 and 2.

Unfortunately there are two objections to be raised against the proof of Theorem A as given in [2]. The first of these is that this proof is based on a lemma whose proof is omitted and which seems much more difficult to prove than is the theorem itself. The second objection is that the proof given for Theorem A is incorrect. It breaks down, for example, when G is the region defined in the complex plane by $0<|z|<2$ and C consists of the points $e^{i\theta}$ where $|\theta| \leqq \frac{3}{2}\pi$.

It will be shown here that, on the basis of a slight increase in the topological background furnished in [2], Theorem A can be proved very simply. After discussing the concepts required we shall, for pedagogical reasons, first prove the most important case:

THEOREM B. *If C is a simple closed curve in the plane and H is a bounded region which contains C together with its interior, then there is a simply connected subregion of H which contains C.*

In the course of the proof we shall establish:

THEOREM C. *A bounded region of the plane is simply connected if its complement is connected.*

Theorem B is sufficient for the immediate applications made of Theorem A in [2]. It will be apparent from its proof that the special case of the latter obtained by restricting G to be bounded can be proved merely by replacing the phrase 'exterior of C' in the proof of Theorem B by 'outer region determined by C,' and replacing 'H' by 'G.' The proof of Theorem A can then be completed by proving:

THEOREM D. *If C is a continuum in the plane and G is a region whose complement is contained in the outer region determined by C then there is a bounded subregion of G which has the same property.*

Finally we shall add some remarks, in section 4, concerning the proof of another statement made in [2].

Although the material presented in the following sections is certainly well known it is hoped that the presentation of it, particularly in sections 2 and 3, below, will make it more easily available for use in courses on the theory of functions of a complex variable.

2. We assume knowledge of the definitions of open set, closed set and boundary, and of the facts that the complements of open (closed) sets are closed (open), that the intersection (join) of finitely many open (closed) sets is open (closed), that the boundary of any set is closed, that any two complementary sets have the same boundary, that an open set is one which does not intersect its boundary, while a closed set is one which contains its boundary, and that the distance between two disjoint closed sets, one of which is bounded, is positive.

The additional topological concepts needed have to do with connectedness. In [2] this notion is defined separately as applied to open sets (arcwise connectedness) and to bounded closed sets (chainwise connectedness). A connected open set is called a *region*, a bounded closed connected set is called a *continuum*. A pair of equations $x=x(t)$, $y=y(t)$, where x and y are continuous functions of t on some interval $a \leqq t \leqq b$, defines a parameterized continuous curve. The set of points (x, y) obtained as t ranges over the interval is called a *continuous curve*. It is easy to prove that a continuous curve is a closed and bounded set, a result often taken for granted in [2]. Using the uniform continuity of x and y it is easy to prove that this set is chainwise connected. Since two nonempty disjoint bounded closed sets have a positive distance it follows that a bounded closed connected set is not the join of two nonempty disjoint closed sets. The converse

is proved as follows: Let p and q be points of the bounded closed but not connected set E such that for some $\epsilon>0$ there is no ϵ-chain of points of E joining p and q. If A is the set of points of E which can be ϵ-chained to p then A and $E-A$ are nonempty disjoint closed sets. (The boundedness of E is not needed for this proof.) It follows from the first of these two results that an open set which has the property that each of its two-point subsets is contained in a continuum contained in it is not the join of two nonempty disjoint open sets. By the method of proof used in establishing the second it is readily shown that in an open set which is not the join of two nonempty disjoint open sets any two points are endpoints of a polygonal arc. (If desired one may add 'each of whose segments is parallel to one of the coordinate axes'). For this one needs the important fact that, for any $\epsilon>0$, the ϵ-neighborhood of any point is arcwise connected. From the first of these two results on open sets it follows that no nonempty proper subset of the plane can be both open and closed, *i.e.* that every nonempty proper subset of the plane has a nonempty boundary.

The theory of connectedness can now be unified by remarking that we have been dealing with two special cases of the definition according to which any set is *connected* if it is not the join of two separated sets, where two sets are called *separated* if they are nonempty, disjoint, and neither contains a limit point of the other. As an immediate consequence of this definition it follows that a connected set which is contained in the join of two separated sets must be wholly contained in one of them, and so the join of the members of any family of connected sets all of which have a common point is connected. This leads to the result that every set is the join of maximal disjoint connected sets, its *components*. (If p is a point of the set E, then the component of E containing p is the join of all the connected subsets of E which contain p.) The components of closed sets are evidently closed and, using the connectedness of ϵ-neighborhoods, the components of open sets are open. Hence the boundary of any component of a closed or open set is contained in the boundary of the set.

Among the components of the complement of a bounded closed set there is manifestly only one which is not bounded. This one is called the *outer region* determined by the set. The Jordan Curve Theorem states that the complement of any simple closed curve in the plane consists of just two regions each of which has the entire curve for its boundary. The bounded one of these is called the *interior* of the curve while the outer region determined by the curve is called its *exterior*. A region is *simply connected* if it contains the interior of every simple closed curve contained in it.

Finally, we need the following theorem, a special case of which was mentioned in connection with the definition of 'component': *If a class of connected sets has a member K such that every member of the class either contains a point of K or has a limit point in K then the join of all members of the class is connected.*

3. Proof of Theorem B: Since C is a bounded closed set and the boundary of H is a closed subset of the exterior of C, the plane can be subdivided, by lines

parallel to the coordinate axes, into equal closed squares so small that any which intersect the boundary of H lie entirely in the exterior of C. It suffices to make sure that the common diagonal length of the squares is less than the distance from the boundary of H to C. For then no square which intersects the former can intersect the latter and, since each square is a connected set, any square which does not intersect C must lie entirely in one component of the complement of C. Since the boundary of H is a bounded set it intersects only finitely many of the squares. Since the exterior of C is a region, these finitely many squares can be joined together by finitely many arcs, each lying in the exterior of C. The result is a continuum K which contains the boundary of H and is contained in the exterior of C. $H-K$ is an open set containing C, and since C is connected it is contained entirely in some one component, H_1, of $H-K$. H_1 is a subregion of H which contains C.

We prove now that the complement of H_1 is connected. This set is the join of K, the complement of H, and the remaining components of $H-K$. Every component of the complement of H contains a point of the boundary of H and hence contains a point of K. Every component of $H-K$ has a boundary point in the boundary of $H-K$ and so has a limit point in K. Hence the complement of H_1 is connected.

It now follows that H_1 is simply connected: Let J be any simple closed curve contained in H_1. The complement of H_1 is connected and, since H_1 is bounded, contains points of the exterior of J. Since it contains no point of J it must be contained entirely in the exterior of J. Hence the interior of J is contained in H_1.

This completes the proof of Theorem B, and the last paragraph contains the proof of Theorem C. The only properties of the exterior of C which have been used are that it is connected and that its boundary is a subset of C. Since the outer region determined by the continuum of Theorem A has these properties there remains only the proof of Theorem D:

Let G_1 be a circular region containing C. Then the complement of G_1 is contained in the outer region determined by C, and the same is true of the complement of the intersection of G and G_1. Let H be the component of this intersection which contains C. H is a bounded subregion of G which contains C, and its complement is the join of the complement of the intersection of G and G_1 and the remaining components of this intersection. But each of the latter has a limit point in the former and, being connected and disjoint from C, is also contained in the outer region determined by C.

4. On p. 57 of [2] Theorem B is used to show that if C and C_1 are rectifiable simple closed curves such that C_1 is interior to C, and if both C and C_1 as well as the region between them lie in the domain of regularity of a function, then the integrals of this function taken around C and C_1 in the positive direction are equal. The proof requires the construction of two "cuts" joining C and C_1, and a footnote describes how this can be done using rays emanating from the interior

of C_1. On the same page it is asserted that the similar theorem in which C_1 is replaced by a finite number of simple closed curves, $C_1, C_2, \cdots, C_n$, each of which is exterior to each of the others, can be proved in a similar manner. However, simple examples show that even with $n=2$ it is not always possible to choose line segments for the cuts required by the proof. Hence the procedure described in the footnote must fail, and the topological foundations of the proof need strengthening. (It is possible to substitute an inductive proof based on the existence of two rectilinear cuts joining C with one of those interior curves which is at minimum distance from it. For this see [5], p. 196.)

To carry out the proof suggested in [2] one can use the

FIRST THEOREM OF JANISZEWSKI: *If E and F are closed subsets of the plane one of which is bounded and whose intersection is connected and if neither E nor F separates p and q then their join does not separate p and q.*

A set E *separates* p *and* q in a connected set M if $M-E$ is the join of two separated sets one of which contains p and the other q. We shall sometimes suppress the reference to M in case M is the entire plane, and shall also say that E *separates* M if it separates some pair of points of M. If G is a region and E is closed, then it follows from material developed in section 2, above, that E separates p and q in G if and only if every arc joining p and q in G intersects E. Also E separates p and q in G if and only if the join of E and the boundary of G separates p and q in the plane. Finally (although we make no use of this) the above theorem can easily be seen to be equivalent to the statement that, G being a region and A a bounded closed set such that $A-G$ is connected, if A separates p and q in G, then A separates p and q in the plane.

From the Theorem of Janiszewski it follows by induction that if $C, C_1, \cdots, C_n$ are as described above, then the set G of all points which are interior to C and exterior to each of the C_i is a region. To carry out the suggested proof we need to be able to construct non-intersecting rectifiable arcs joining C and C_1, C_1 and $C_2, \cdots, C_n$ and C, such that each lies, except for its end points, in G. The first step of the construction is taken care of by

THEOREM E: *If p and q are boundary points of a region G in the plane and $\epsilon>0$ then there exist boundary points p_1 and q_1 at a distance less than ϵ from p and q, respectively, such that there is a polygonal arc joining p_1 and q_1 which, except for its end points, lies in G.*

Proof: It is sufficient to choose p_0 and q_0 in G and within a distance ϵ of p and q, respectively, and to join p to p_0 by a segment, p_0 to q_0 by a polygonal arc in G, and q_0 to q by a segment. If p_1 is the first boundary point of G on the segment from p_0 to p, q_1 is the first boundary point of G on the segment from q_0 to q, and, as may be assumed, ϵ is at most half the distance from p to q, then it is easy to pick out the desired polygonal arc. (If the boundary of G is a simple closed curve then p_1 and q_1 may be chosen to be the given points p and q, but this is somewhat more difficult to prove and is not necessary for our purposes.)

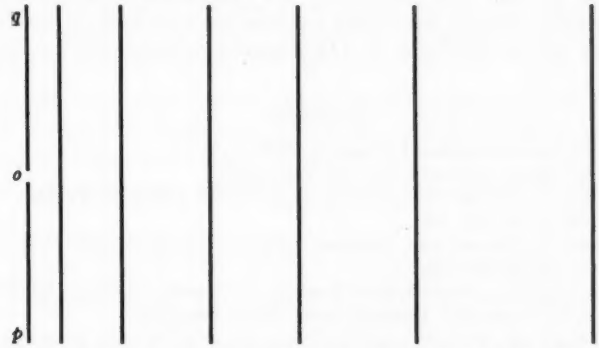
In order to complete the construction it would clearly be sufficient to know

THEOREM F: *An arc which lies, except for its endpoints, in a region G whose boundary is a bounded set does not separate G unless its endpoints lie in the same component of the boundary of G.*

To establish this we need two lemmas whose proofs may be found, for example, in [5].

LEMMA 1: *If p and q belong to different components of the bounded closed set E in the plane then E is the join of two separated sets one of which contains p and the other q.*

LEMMA 2: *No arc separates the plane.*



The example of a set E, suggested by the figure, which consists of two half-open segments $[p, o)$ and $[q, o)$ and a sequence of closed segments having $[p, q]$ as limit set, shows that the word 'closed' cannot be omitted from the hypothesis of Lemma 1. The set obtained from this one by subjecting it to an inversion with o as center shows that 'bounded' cannot be omitted.

Lemma 2 is easily proved by using Janiszewski's theorem. For from this it follows that if some arc separates two points, then one of its two "halves" also does. Iteration of this argument leads to a contradiction.

Lemma 1 can be avoided if, in the hypothesis of Theorem F, 'is a bounded set' is replaced by 'has finitely many components.' This weaker form of the theorem is sufficient for our present needs.

Proof of Theorem F: Let p and q be points of G not on the arc K. If at most one endpoint of K is on the boundary of G then, since the latter does not separate p and q, it follows from Lemma 2 and the Theorem of Janiszewski that the join of K and the boundary of G does not separate p and q. By a previous remark it follows that K does not separate p and q in G. (In this part of the proof the condition that the boundary of G is a bounded set is not needed.)

If both endpoints of K are on the boundary of G, but lie in different components of this boundary, then the latter is the join of two separated sets, S_1 and S_2,

such that one endpoint of K is in S_1 and the other is in S_2. None of the sets S_1, S_2, K separates p and q so, by the Theorem of Janiszewski, neither the join of S_1 and K nor the join of S_2 and K separates p and q. Appealing once more to the same theorem it follows from this that the join of K and the boundary of G does not separate p and q. Hence K does not separate p and q in G.

5. Using Theorems E and F one can also prove that there exists a polygonal arc which lies, except for its endpoints, in the region G described in the third paragraph of the preceding section and which separates two of the curves C_i in G. This can be used as a basis for another inductive proof of the theorem of [2] which is in question.

For a discussion of the Theorem of Janiszewski (including a proof of the closely related Jordan Curve Theorem) the reader is referred to [3] and [4]. For a proof of the theorem itself (and a connected treatment of the topology of the plane) see [5], in particular p. 112. Related material will be found in [6], Chapter VI.

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APPLICATIONS OF THE ELEVEN POINT CONIC

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1. Introduction to the idea of plane at infinity. The equation of a sphere in homogeneous cartesian coordinates is

$$x^2 + y^2 + z^2 + 2uxt + 2vyt + 2wzt + dt^2 = 0.$$

Its section by the plane at infinity whose equation is $t=0$ is

$$x^2 + y^2 + z^2 = 0.$$

This curve whose equations are

$$t = 0, \qquad x^2 + y^2 + z^2 = 0,$$

is called the circle on the plane at infinity. We will denote it by Ω.

The section of a surface by the plane at infinity is called its trace on the plane at infinity. It can be easily seen that the trace of a quadric on the plane at infinity is a conic. If the equation of the quadric is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2 = 0,$$

then equations of the trace are

$$t = 0, \qquad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

In what follows we will write only 'trace' instead of 'trace on the plane at infinity.'

Again it can be proved easily that:

(i) The traces of two perpendicular lines are two points conjugate w.r.t. Ω.

(ii) The traces of the two axes of any plane section of a quadric (having a unique center in finite space) are two points conjugate w.r.t. Ω as well as q, where q denotes the trace of the quadric.

2. Definition and equation of eleven point conic. Let S and S' be two conics. Let P and P' be two points conjugate w.r.t. S and S'. If P traces out a straight line, P' traces out a conic.

Proof. In homogeneous coordinates let the equations of S and S' be respectively:

$$S(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$S'(x, y, z) \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0.$$

Let P be (α, β, γ) and P' (ξ, η, ζ). Let the locus of P be the line:

$$lx + my + nz = 0. \tag{1}$$

Equations of the polars of P' w.r.t. S and S' are respectively,

$$x\frac{\partial S}{\partial \xi} + y\frac{\partial S}{\partial \eta} + z\frac{\partial S}{\partial \zeta} = 0 \tag{2}$$

and

$$x\frac{\partial S'}{\partial \xi} + y\frac{\partial S'}{\partial \eta} + z\frac{\partial S'}{\partial \zeta} = 0. \tag{3}$$

P lies on 1, 2, and 3; therefore

$$\begin{vmatrix} l & m & n \\ \dfrac{\partial S}{\partial \xi} & \dfrac{\partial S}{\partial \eta} & \dfrac{\partial S}{\partial \zeta} \\ \dfrac{\partial S'}{\partial \xi} & \dfrac{\partial S'}{\partial \eta} & \dfrac{\partial S'}{\partial \zeta} \end{vmatrix} = 0.$$

Hence the locus of P' is

$$\begin{vmatrix} l & m & n \\ \dfrac{\partial S}{\partial x} & \dfrac{\partial S}{\partial y} & \dfrac{\partial S}{\partial z} \\ \dfrac{\partial S'}{\partial x} & \dfrac{\partial S'}{\partial y} & \dfrac{\partial S'}{\partial z} \end{vmatrix} = 0,$$

which is a conic. This is called the 'Eleven Point Conic' of the family (of conics) $S+\lambda S'=0$ corresponding to the locus of P, where λ is a parameter. It is called the eleven point conic because it passes through eleven conspicuous points which can be constructed at once from the locus of P and the quadrangle whose vertices are the four points of intersection of S and S'. As there is hardly any need of specifying these eleven points here, we leave it at that.

3. Applications.

First. The axes of the section of the quadric

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

by the plane $lx+my+nz=0$ *are lines in which the plane cuts the cone*

$$(mz - ny)\frac{\partial f}{\partial x} + (nx - lz)\frac{\partial f}{\partial y} + (ly - mx)\frac{\partial f}{\partial z} = 0.$$

Let A and B be the traces of the axes of the section of the quadric by the given plane. Then A and B lie on the trace of the plane; *i.e.*,

$$lx + my + nz = 0, \qquad t = 0.$$

Let q be the trace of the quadric, then its equations are

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad t = 0,$$

or

$$f(x, y, z) = 0, \qquad t = 0.$$

The equations of the eleven point conic of the family $q+\lambda\Omega=0$, corresponding to the line $lx+my+nz=0$, $t=0$ are

$$t = 0, \quad \begin{vmatrix} l & m & n \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ x & y & z \end{vmatrix} = 0.$$

As the points A and B are conjugate w.r.t. q and Ω, they lie on this eleven point conic, by definition. Now the axes are, obviously, OA and OB where O

is the origin. Hence OA and OB lie on the cone

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ l & m & n \\ x & y & z \end{vmatrix} = 0$$

or

$$(mz - ny)\frac{\partial f}{\partial x} + (nx - lz)\frac{\partial f}{\partial y} + (ly - mx)\frac{\partial f}{\partial z} = 0.$$

Since OA and OB lie on the given plane also, they are lines of intersection of this cone and the plane.

COROLLARY. *If the area of a central section of the ellipsoid* $x^2/a^2+y^2/b^2+z^2/c^2=1$ *is constant, then the axes of the section lie on the cone*

$$\sum \frac{a^2 - p^2}{a^4}\left[\frac{c^2 - a^2}{c^2}\frac{z}{y} - \frac{a^2 - b^2}{b^2}\frac{y}{z}\right]^2 = 0,$$

where p *is the distance from the center of the tangent plane parallel to any of the planes of the section.*

Let $lx+my+nz=0$ be such a central section. If α and β are the semi-axes of the section, then

$$\alpha\beta = abc\sqrt{\frac{l^2 + m^2 + n^2}{a^2l^2 + b^2m^2 + c^2n^2}} = \frac{abc}{p}.$$

As the area is constant, p is constant. Again

$$a^2l^2 + b^2m^2 + c^2n^2 = p^2(l^2 + m^2 + n^2). \tag{4}$$

Let

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$$

be the equation of an axis of the section. As it lies on the plane of section,

$$ll_1 + mm_1 + nn_1 = 0. \tag{5}$$

Again it lies on the cone

$$(mz - ny)\frac{x}{a^2} + (nx - lz)\frac{y}{b^2} + (ly - mx)\frac{z}{c^2} = 0.$$

Therefore

$$\sum \frac{l}{l_1} a^2(b^2 - c^2) = 0. \tag{6}$$

Solving 5 and 6 simultaneously, we get

$$\frac{l}{\frac{n_1}{m_1} b^2(c^2 - a^2) - \frac{m_1}{n_1} c^2(a^2 - b^2)} = \frac{m}{\text{etc.}} = \frac{n}{\text{etc.}}$$

or

$$\frac{l}{\frac{n_1}{m_1}\frac{c^2 - a^2}{c^2a^2} - \frac{m_1}{n_1}\frac{a^2 - b^2}{a^2b^2}} = \frac{m}{\text{etc.}} = \frac{n}{\text{etc.}}.$$

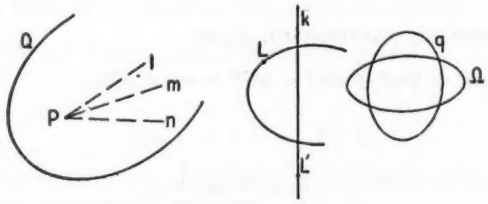
Substituting values of l, m, n from these equations in 4, we get

$$\sum \frac{a^2 - p^2}{a^4}\left[\frac{c^2 - a^2}{c^2}\frac{n_1}{m_1} - \frac{a^2 - b^2}{b^2}\frac{m_1}{n_1}\right]^2 = 0,$$

which is the condition that the axis $x/l_1 = y/m_1 = z/n_1$ should lie on the cone

$$\sum \frac{a^2 - p^2}{a^4}\left[\frac{c^2 - a^2}{c^2}\frac{z}{y} - \frac{a^2 - b^2}{b^2}\frac{y}{z}\right]^2 = 0.$$

Second. If P is the center of a section of a central quadric, then the axes of the section are the straight lines in which the plane of section cuts the cone containing the normals from P.

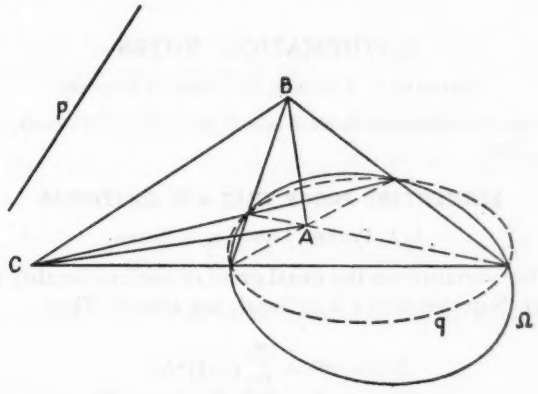


Let Q be the given quadric. Let us find out the locus of the lines through P which are at right angles to their polar lines. Let $l, m, n, \cdots$, be such lines through P. Let the trace of the polar plane of P w.r.t. Q be the line k. As polar lines of $l, m, n, \cdots$ lie on the polar plane of P, the traces of the polar lines lie on k. Let the traces of l and its polar line be L and L' respectively. Now L and L' are conjugate points w.r.t. q and Ω. As the locus of L' is the line k, the locus of L is the eleven point conic of q and Ω corresponding to k. Hence lines $l, m, n, \cdots$ lie on the cone which is formed by the lines joining P to this eleven point conic.

As the normals from P are also at right angles to their polar lines, they lie on this cone and for this reason the cone is called the cone containing the normals.

Now the plane section of which P is the center is the plane through P parallel to its polar plane, *i.e.*, the plane through P and k. Let us show that the points of intersection of k and the eleven point conic are the traces of the axes of this plane section. According to Desargues' theorem line k will cut the conics of the family $q+\lambda\Omega=0$ in pairs of points which belong to the same involution range and the double points of this range are the two points on k conjugate w.r.t. every member of the family $q+\lambda\Omega=0$. These double points, by definition, lie on the eleven point conic. Hence k cuts the eleven point conic in two points which are conjugate w.r.t. q and Ω. Thus lines joining P to these points are axes in the section of which P is the center. These axes obviously lie on the cone containing normals from P to Q.

Third. If q is the trace of a central quadric Q, then a cone C whose vertex is the centre of Q and whose trace is any eleven point conic of the family $q+\lambda\Omega=0$ passes through an unlimited number of sets of conjugate diameters of Q and possesses innumerable sets of three mutually perpendicular generators.



Let triangle ABC be the unique common self conjugate triangle of q and Ω. Now any eleven point conic of the family $q+\lambda\Omega=0$, say the one corresponding to any line p, passes through the vertices of the triangle ABC. For A is conjugate to the point in which BC cuts p; and so on. Let us call this eleven point conic e.

As e circumscribes a triangle self-conjugate w.r.t. q and Ω, e is harmonically circumscribed to q and Ω. Therefore there exist:

(i) innumerable triangles inscribed in e and self-conjugate w.r.t. q, and,

(ii) innumerable triangles inscribed in e and self-conjugate w.r.t. Ω.

If we join the vertices of any triangle from (i) to the center of Q, we get a set of conjugate diameters of Q. Hence C passes through an unlimited number of

sets of conjugate diameters of Q.

Again if we join the vertices of any triangle from (ii) to the centre of Q, we get a set of three mutually perpendicular lines.

COROLLARY 1. *C passes through the three principal axes of Q.* For e passes through A, B, C which are the traces of principal axes.

COROLLARY 2. *If P is the central plane of which p is the trace then the normal to P at the center and the diameter of which P is the diametral plane lie on 'C.'*

From the definition it can be proved that the pole of p w.r.t. any conic of the family $q+\lambda\Omega=0$ lies on e. As the pole of p w.r.t. Ω is the trace of any normal to P, the normal at the center lies on C. Again the pole of p w.r.t. q, which must lie on e, is the trace of the diameter D of which P is the diametral plane. Hence D must lie on C.

MATHEMATICAL NOTES

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ELEMENTARY PROOF THAT e IS IRRATIONAL

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The following variation on the usual proof of the irrationality of e is perhaps slightly simpler. Suppose that e is rational, say $e=a/b$. Then

$$b/a = e^{-1} = \sum_{n=0}^{\infty} (-1)^n/n!,$$

and multiplication by $(-1)^{a+1}a!$ and transposition of terms gives

$$(-1)^{a+1}\left\{b(a-1)! - \sum_{n=0}^{a} (-1)^n a!/n!\right\}$$
$$= \frac{1}{(a+1)} - \frac{1}{(a+1)(a+2)} + \frac{1}{(a+1)(a+2)(a+3)} - \cdots.$$

The right side has a value between 0 and 1 since the alternating series clearly converges to a value between its first term and the sum of its first two terms. But the left side is an integer, so we have a contradiction.

SOME SUMS CONTAINING BERNOULLI FUNCTIONS

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According to Problem 4375, proposed by N. J. Fine (solution by T. M. Apostol, this MONTHLY, vol. 58, 1951, p. 426), the sums

$$\sum_{r=0}^{k-1} ((2^r x + \tfrac{1}{2})) \tag{1}$$

are uniformly bounded (with bound = 1); here $((x)) = x - [x] - \frac{1}{2}$ and x is real.

Let the Bernoulli polynomial $B_m(x)$ be defined by means of

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} B_m(x) = \frac{te^{xt}}{e^t - 1},$$

and let $\overline{B}_m(x)$ denote the corresponding Bernoulli function; that is $\overline{B}_m(x)$ is periodic with period 1 and coincides with $B_m(x)$ for $0 \leqq x < 1$. Since $\overline{B}_1(x) = ((x))$ it is natural to consider the sum

$$\sum_{r=0}^{k-1} 2^{r(1-m)} \overline{B}_m(2^r x + \tfrac{1}{2}), \tag{2}$$

which reduces to (1) for $m = 1$.

Now in the well-known formula [3, 21]

$$2^{1-m} B_m(2x) = B_m(x) + B_m(x + \tfrac{1}{2}), \tag{3}$$

replace x by $2^r x$, $r = 0, \cdots, k-1$, multiply by $2^{r(1-m)}$ and add the resulting equations. We find that

$$\sum_{r=0}^{k-1} 2^{r(1-m)} B_m(2^r x + \tfrac{1}{2}) = 2^{k(1-m)} B_m(2^k x) - B_m(x). \tag{4}$$

This equation remains valid if $B_m(x)$ is replaced by $\overline{B}_m(x)$. Thus for example we find that

$$\sum_{r=0}^{\infty} 2^{r(1-2m)} \overline{B}_m(2^r x + \tfrac{1}{2}) = -\overline{B}_m(x), \qquad (m > 1).$$

By means of (4) we can find a bound for the sum (2). Lehmer [2] has discussed the maximum and minimum values of $\overline{B}_m(x)$; for example,

$$|\overline{B}_{2m}(x)| \leqq |B_{2m}|, \tag{5}$$

where $B_{2m} = B_{2m}(0)$. The corresponding result for odd index is more complicated. Substituting from (5) in (4) we get

$$\left| \sum_{r=0}^{k-1} 2^{r(1-2m)} B_{2m}(2^r x + \tfrac{1}{2}) \right| \leqq 2 |B_{2m}|, \qquad (m > 0).$$

If in place of (3) we use the more general multiplication formula [3, 21]

$$h^{1-m}B_m(hk) = \sum_{s=0}^{h-1} B_m\left(x + \frac{s}{h}\right),$$

which is valid for $\overline{B}_m(x)$ also, we find that

$$\sum_{s=1}^{h-1}\sum_{r=0}^{k-1} h^{r(1-m)}B_m\left(h^r x + \frac{s}{h}\right) = h^{k(1-m)}B_m(h^k x) - B_m(x).$$

In particular

$$\sum_{s=1}^{h-1}\sum_{r=0}^{\infty} h^{r(1-m)}\overline{B}_m\left(h^r x + \frac{s}{h}\right) = -\overline{B}_m(x), \qquad (m > 1).$$

Corresponding results may be obtained for the Euler polynomials. However it is perhaps more instructive to consider the following more general situation. Following Euler [1, 487–491] we define $\eta_m(x, \alpha)$ by means of

$$\sum_{m=0}^{\infty} \frac{t^m}{m!}\eta_m(x, \alpha) = \frac{(1-\alpha)e^{xt}}{1-\alpha e^t}, \qquad (\alpha \neq 1); \tag{6}$$

for $\alpha = -1$ it is evident that $\eta_m(x, -1) = E_m(x)$, the Euler polynomial. It may be verified, using (6), that

$$h^m \sum_{s=0}^{h-1} \alpha^s \eta_m\left(x + \frac{s}{h}, \alpha\right) = \eta_m(hx, \alpha), \tag{7}$$

provided $\alpha^{h-1} = 1$. Then as above it follows from (7) that

$$\sum_{s=1}^{h-1} \alpha^s \sum_{r=0}^{k-1} h^{-rm}\eta_m\left(h^r x + \frac{s}{h}, \alpha\right) = h^{-km}\eta_m(h^k x, \alpha) - \eta_m(x, \alpha). \tag{8}$$

If next we define $\bar{\eta}_m(x, \alpha)$ by means of

$$\bar{\eta}_m(x, \alpha) = \eta_m(x, \alpha), \qquad (0 \leqq x < 1),$$
$$\bar{\eta}_m(x+1, \alpha) = \alpha^{-1}\bar{\eta}_m(x, \alpha),$$

then (7) and (8) hold for $\bar{\eta}_m(x, \alpha)$ and we find that

$$\sum_{s=1}^{h-1} \alpha^s \sum_{r=0}^{\infty} h^{-rm}\bar{\eta}_m\left(h^r x + \frac{s}{h}, \alpha\right) = -\bar{\eta}_m(x, \alpha), \qquad (m > 0).$$

References

1. L. Euler, Institutiones calculi differentialis, Petrograd, 1755.

2. D. H. Lehmer, On the maxima and minima of Bernoulli polynomials, this MONTHLY, vol. 47, 1940, pp. 533–538.

3. N. E. Nörlund, Differenzenrechnung, Berlin, 1924.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

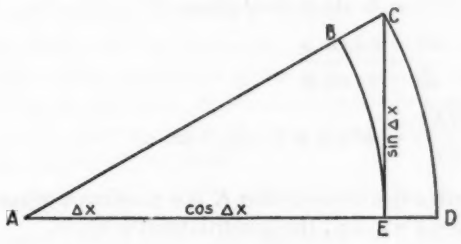
All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

LIMITS USED IN THE DERIVATIVE OF sin x

D. K. PEASE, University of Connecticut at Hartford

When presenting the derivative of sin x, students are often particularly disturbed at two points. First, when we want $\lim_{\Delta x \to 0} (\sin \Delta x)/\Delta x$ we proceed to find $\lim_{\Delta x \to 0} \Delta x/\sin \Delta x$ which they grudgingly admit is all right, but they ask "Why not find $\lim_{\Delta x \to 0} (\sin \Delta x)/\Delta x$ directly?" Second, when we evaluate $\lim_{\Delta x \to 0} (1-\cos \Delta x)/\Delta x$ we use a complicated and unfamiliar half-angle formula. The following presentation circumvents these particular difficulties.

To find $\lim_{\Delta x \to 0} (\sin \Delta x)/\Delta x$ consider the following diagram with $AC=1$.



We have sector $ABE <$ triangle $ACE <$ sector ACD,

$$(\Delta x \cos^2 \Delta x)/2 < (\sin \Delta x \cos \Delta x)/2 < \Delta x/2,$$

$$\cos \Delta x < (\sin \Delta x)/\Delta x < 1/\cos \Delta x,$$

and the conclusion is evident.

To find $\lim_{\Delta x \to 0}(1-\cos \Delta x)/\Delta x$ multiply by

$$\frac{1+\cos \Delta x}{1+\cos \Delta x}$$

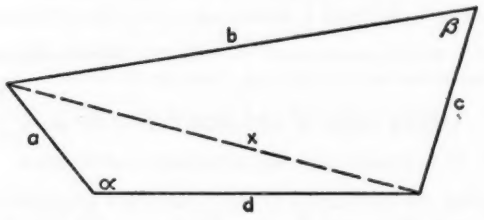
to get

$$\frac{1-\cos^2 \Delta x}{\Delta x(1+\cos \Delta x)} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{\sin \Delta x}{1+\cos \Delta x}$$

and the limit is evident.

COMMENTS ON QUADRILATERALS

L. A. RINGENBERG, Eastern Illinois State College



Comment 1. Given a quadrilateral with fixed sides a, b, c, d, in a fixed order, the area of the quadrilateral is a maximum when the quadrilateral is cyclic.

Proof. Let K denote the area of the quadrilateral. In terms of the notation of the figure we have

$$x^2 = b^2 + c^2 - 2bc\cos\beta = a^2 + d^2 - 2ad\cos\alpha$$

$$2K = bc\sin\beta + ad\sin\alpha$$

$$\frac{d\beta}{d\alpha} = \frac{ad\sin\alpha}{bc\sin\beta}$$

$$\frac{d(2K)}{d\alpha} = ad(\cot\alpha + \cot\beta)\sin\alpha.$$

The second derivative test reveals that K is a maximum when $\cot\alpha+\cot\beta=0$; this occurs when $\alpha+\beta=\pi$, *i.e.*, the quadrilateral is cyclic.

Comment 2. The area of a cyclic quadrilateral with sides a, b, c, d, in that order is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. Note that Heron's formula for the area of a triangle results if a, b, c remain fixed and d approaches 0.

Proof. From proof of comment 1,

$$K = \tfrac{1}{2}(ad + bc)\sin\alpha,\ \cos\alpha = \frac{a^2 + d^2 - b^2 - c^2}{2ad + 2bc},$$

$$K^2 = \frac{(bc + ad)^2}{4}\cdot\frac{(2bc + 2ad)^2 - (a^2 + d^2 - b^2 - c^2)^2}{4(bc + ad)^2},$$

$$K^2 = \left(\frac{b + c + d - a}{2}\right)\left(\frac{a + c + d - b}{2}\right)\left(\frac{a + b + d - c}{2}\right)\left(\frac{a + b + c - d}{2}\right),$$

which reduces easily to the formula given above.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1048 [1953, 40]. *Correction.*

Let M be an $n\times\infty$ matrix. Show that if every $n\times n$ submatrix of M is singular, then the *rows* of M are linearly dependent.

E 1076. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

John has a one dollar bill and wishes to change it into cold hard cash, any combination of coins, even perhaps a silver dollar. Jacob has a \$100 bill and wants it changed into smaller bills; he wants at least five ones; he doesn't want all ones and he doesn't want any twos. In how many different ways can a bank teller make change for John? for Jacob?

E 1077. *Proposed by J. A. Ward, University of Kentucky*

Let b be an integer greater than 2 and let $n=b^{b-1}-1$. Show that

(1) $n/(b-1)$ when converted to base b is a $(b-1)$-digit integer whose digits are all ones,

(2) $n/(b-1)^2$ when converted to base b is a $(b-2)$-digit integer whose digits from left to right are $1, 2, 3, \cdots, b-4, b-3, b-1$.

E 1078. *Proposed by N. Shklov, University of Saskatchewan*

Let a real positive number n be split into x equal parts in such a manner that the product of the parts will be greatest. How many parts will there be?

E 1079. *Proposed by V. L. Klee, Jr., University of Virginia*

Suppose f and g are real functions such that always $g(x-y)=g(x)g(y)+f(x)f(y)$, and such that $f(t)=1$ and $g(t)=0$ for some $t\neq 0$. Prove that always $g(x\pm y)=g(x)g(y)\mp f(x)f(y)$ and $f(x\pm y)=f(x)g(y)\pm g(x)f(y)$.

E 1080. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let I be the incenter, N the nine-point center, and D the midpoint of side BC of a triangle ABC. Show that one of the common tangents to the circles $I(N)$ and $D(N)$ is parallel to BC.

SOLUTIONS

The Equation $d^2x/dy^2 = 1/(d^2y/dx^2)$

E 1046 [1953, 40]. *Proposed by E. P. Starke, Rutgers University*

Knowing that dy/dx and dx/dy are reciprocals, students sometimes jump to the conclusion that d^2x/dy^2 should be the reciprocal of d^2y/dx^2. Determine functions $y=f(x)$ for which this is indeed true.

Solution by Arthur Rosenthal, Purdue University. From $dx/dy = 1/(dy/dx)$ there results $d^2x/dy^2 = -(d^2y/dx^2)/(dy/dx)^3$. Hence $d^2x/dy^2 = 1/(d^2y/dx^2)$ yields $(d^2y/dx^2)^2 = -(dy/dx)^3$, or $dy'/dx = (-y')^{3/2}$. Therefore $y' = -4/(x+a)^2$, whence $y = 4/(x+a) - b$, which is easily shown to verify the original differential equation. In symmetric form we have $(x+a)(y+b) = 4$, the family of all translations of the rectangular hyperbola $xy = 4$.

Also solved by P. M. Anselone, Louis Berkofsky, Julian Braun, F. M. Carpenter, A. E. Danese, Fred Discepoli, I. A. Dodes, A. L. Epstein, H. M. Feldman, B. A. Fleishman, Lloyd Fulk, Harry Furstenberg, Emil Grosswald, Frank Herlihy, M. H. Hoehn, Vern Hoggatt, R. Huck, A. R. Hyde, M. S. Klamkin, R. B. Leckie, H. L. Lee, T. Lesser, Samuel Leshnover, E. W. Marchand, E. P. Miles, Jr., D. G. Miller, George Millman, M. Morduchow, S. Parameswaran, L. L. Pennisi, B. E. Rhoades, C. C. Richtmeyer, L. A. Ringenberg, Azriel Rosenfeld, W. G. Rouleau, Charles Salkind, L. E. Sessions, II, A. Sisk, H. W. Smith, O. E. Stanaitis, D. D. Strebe, Donato Teodoro, J. V. Whittaker, J. W. Young, David Zeitlin, and the Proposer. Late solution by John Burke.

Parameswaran pointed out that, in general, d^ny/dx^n and d^nx/dy^n are reciprocals if $(x+a)(y+b) = (n!)^{2/(n-1)}$.

Gaussian Factors

E 1047 [1953, 40]. *Proposed by W. R. Ransom, Tufts College*

Find all Gaussian integral factorizations of $10i$ where $i = \sqrt{-1}$.

Solution by Marilyn Marshall, University of North Carolina. We want to obtain the essentially different factorizations, those which cannot be obtained from each other by multiplying each factor by a unit such that the product of these units for one factorization equals 1. The units are 1, -1, i, and $-i$, and the unique prime factorization of $10i$ is

$$10i = (1+i)^2(2+i)(2-i).$$

Including the trivial factorization, $(1)(10i)$, we obtain six different factorizations containing two factors, four different factorizations containing three factors, and the one factorization containing the four factors. Therefore we have 11 essentially different factorizations, including the trivial one. They are: $(1)(10i)$, $(1+i)(5+5i)$, $(2i)(5)$, $(2+i)(2+4i)$, $(2-i)(-2-4i)$, $(1+3i)(3+i)$, $(2i)(2+i)(2-1)$, $(5)(1+i)(1+i)$, $(1+3i)(1+i)(2-i)$, $(3+i)(1+i)(2+i)$, $(1+i)\cdot(1+i)(2+i)(2-i)$.

Also solved by A. N. Aheart, R. H. Boyer, W. E. Briggs, Fred Discepoli, I. A. Dodes, M. S. Klamkin, G. W. Lofquist, T. F. Mulcrone, L. L. Pennisi, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, J. H. Skelton, William Small, C. W. Trigg, R. E. Wild, and the Proposer. Many of these solutions were not complete.

Center of Gravity of a Vertical Rod

E 1049 [1953, 40]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If a uniform thin rod of length L is perpendicular to and touching the earth's surface at one end, find the distance between the center of gravity and the center of mass of the rod.

Solution by M. Morduchow, Polytechnic Institute of Brooklyn. Let x be distance measured along the rod from the earth's surface. Also, let R be the radius and M the mass of the earth, while ρ is the mass per unit length of the rod. Then, by Newton's inverse-square law, the total force F (directed toward the earth's center) acting on the rod will be

$$F = \int_0^L GM\rho(R + x)^{-2}dx = GMm/R(R + L),$$

where G is the gravitational constant, and $m=\rho L$ is the mass of the rod. If the center of gravity is at $x=x_g$, then by definition we require that $F=GMm/(R+x_g)^2$. Therefore $(R+x_g)^2=R(R+L)$, whence

$$x_g = \sqrt{R(R + L)} - R.$$

Since the center of mass is at $x=x_m=L/2$, the center of gravity will be below the center of mass, and the distance between the centers will be

$$x_m - x_g = L/2 - R[(1 + L/R)^{1/2} - 1].$$

If $L/R \ll 1$, then expansion by the binomial theorem to second powers of L/R yields the approximate result

$$x_m - x_g \approx (1/8)(L/R)L.$$

Also solved by Julian Braun, A. L. Epstein, Frank Herlihy, P. F. Hultquist, A. R. Hyde, T. Leser, D. M. Mandelbaum, L. V. Mead, C. S. Ogilvy, Azriel Rosenfeld, H. W. Smith, J. V. Whittaker, R. H. Wilson, Jr., and the Proposer.

All of these solutions were not the same. Braun, Hyde, Mead, Whittaker, and the Proposer took the distance x_g to be

$$x_g = \frac{\int_0^L x(R + x)^{-2}dx}{\int_0^L (R + x)^{-2}dx} = (R/L)(R + L)\ln(1 + L/R) - R.$$

Rosenfeld took the center of gravity of the rod as that point for which the weight of the portion of the rod above the point is equal to the weight of the portion of the rod below the point. He found

$$x_0 = 2RL/(4R + L).$$

All other solutions were like Morduchow's. Here, for a rod a mile long, the distance between the two centers is about 2 inches.

Smith called attention to Kellogg, *Foundations of Potential Theory*, p. 4.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4548. *Proposed by Joel Pitcairn and Robert Ellis, University of Pennsylvania*

Let X be a topological space with the property that every one-to-one transformation of X onto itself is continuous. Characterize the open subsets of X.

4549. *Proposed by Richard Obláth, Budapest, Hungary*

The Gauss-Newton line of the complete quadrilateral formed by the four Feuerbach tangents of a triangle is the Euler line of the triangle.

4550. *Proposed by P. L. Chessin, Cooper Union, New York City*

I. The smallest consecutive cubes whose difference is a triangular number are 6^3 and 5^3. What are the next two pairs?

II. Under the assumption in I, show that this difference is equal to the difference of two squares of the form $(9a+1)^2-9a^2$.

To Victor Thébault, this problem is humbly dedicated.

4551. *Proposed by K.-F. Moppert, Basle, Switzerland*

For $k=0, 1, 2, \cdots$, prove

$$\lim_{n\to\infty} n^{\alpha-k} \sum_{\nu=0}^{n-1} (-1)^\nu \nu^k \binom{\alpha}{\nu} = \frac{1}{(k-\alpha)\Gamma(-\alpha)}.$$

Remark: for $\alpha=-1$ this gives a well known asymptotic value for the sum of the kth powers.

4552. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

What derangement of terms of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

will produce a sum which is rational?

SOLUTIONS

4486 [1952, 254]. *Correction*

In the June-July issue, p. 427, in line 6 of the Solution replace $F(x)$ by $f(x)$, and in line 7 replace $f(z)$ by $F(z)$.

Hypercube and Its Boundaries

4488 [1952, 332]. *Proposed by George Grossman, De Witt Clinton High School, New York City*

Using the symbolic representation

$$c^{(3)} \equiv 8m_0 + 12m_1 + 6m_2 + 1m_3$$

to indicate that a cube of dimensionality 3 possesses 8 vertices (dimensionality 0), 12 manifolds of dimensionality one, *etc.*, then a tesseract (hypercube of dimensionality four) can be represented symbolically by

$$c^{(4)} \equiv 16m_0 + 32m_1 + 24m_2 + 8m_3 + 1m_4.$$

If a hypercube of dimensionality n is represented by

$$c^{(n)} \equiv a_{n0}m_0 + a_{n1}m_1 + \cdots + a_{nn}m_n,$$

determine the coefficients a_{ij}.

I. *Solution by Elizabeth M. Cooper, Hunter College High School, New York City.* If the hypercube $C^{(n)}$ is defined as generated by a $C^{(n-1)}$ moving through an nth dimension, the bounding m_j's fall under the same definition, and the m_j's of $C^{(n+1)}$ evolve from the bounding spaces of $C^{(n)}$ in two and only two ways, namely:

(a) Each m_j of $C^{(n)}$, in its initial and terminal positions, becomes two bounding m_j's of $C^{(n+1)}$.

(b) Each m_{j-1} of $C^{(n)}$ generates a bounding m_j of $C^{(n+1)}$.

This means that

$$a_{n+1,j} = 2a_{nj} + a_{n,j-1}. \tag{1}$$

Since, for $n = 0, 1, 2, 3, 4$,

$$a_{nj} = \binom{n}{j} 2^{n-j}, \tag{2}$$

it follows, by (1), that, for these values of n,

$$a_{n+1,j} = 2\binom{n}{j} 2^{n-j} + \binom{n}{j-1} 2^{n-j+1} = \binom{n+1}{j} 2^{n-j+1}. \tag{3}$$

Hence, by induction, (2) holds for all integral values of n.

II. *Solution by Louis Weiner, DePaul University.* In n-dimensional Euclidean space a hypercube of dimensionality n is formed by the $(n-1)$-dimensional hyperplanes $x_1 = \pm a$, $x_2 = \pm a$, $\cdots$, $x_n = \pm a$. A manifold of dimensionality r $(r = 0, 1, \cdots, n-1)$ is formed by the intersection of $n-r$ of these hyperplanes. We consider $n-r$ hyperplanes

$$x_{i_1} = \epsilon_1 a, \; x_{i_2} = \epsilon_2 a, \; \cdots, \; x_{i_{n-r}} = \epsilon_{n-r} a,$$

where $\epsilon_j = +1$ or -1 and the i_j are all distinct (since we do not consider the intersection of a hyperplane with itself, and two parallel hyperplanes do not intersect). There are $C^n_{n-r} = C^n_r$ possible ways of choosing the subscripts i_j, and for each of these ways there are 2^{n-r} possible combinations of the ϵ_j. There are, therefore, $2^{n-r}(C^n_r)$ manifolds of dimension r $(r = 0, 1, \cdots, n-1)$. Since $a_{nn} = 1$, we have in general

$$a_{nr} = 2^{n-r} C^n_r.$$

Also solved by A. G. Anderson, J. D. Baum, J. H. Braun, W. B. Carver, H. S. M. Coxeter, I. A. Dodes, H. H. Downing, F. G. Fender, Herta T. Freitag, A. M. Glicksman, H. W. Gould, P. K. Henney, B. H. Henry, Norman Johnson, M. S. Klamkin, David Mandelbaum, Martin Pearl, M. Perisastri, Judith A. Richman, L. A. Ringenberg, A. I. Rosenfeld, J. A. Schumaker, S. B. Townes, Alan Wayne, and the Proposer.

Editorial Note. The following pertinent references were supplied by various contributors: Courant and Robbins, *What Is Mathematics?*, p. 230; D. M. Y. Somerville, *Geometry of n Dimensions*, p. 29; H. S. M. Coxeter, *Regular Polytopes*, pp. 122, 128; D. F. Höppner, *Proceedings of the Edinburgh Mathematical Society*, 1896, p. 121.

A Summation

4489 [1952, 332]. *Proposed by D. J. Newman, Harvard University*

Sum the series

$$\sum_{n=1}^{\infty} \frac{(n/e)^{n-1}}{n!}.$$

I. *Solution by H. F. Sandham, Trinity College, Ireland.* Expanding by the rule of partial fractions, we have

$$\frac{1}{1+x}+\frac{x}{(1+x)(2+x)}+\frac{x^2}{(1+x)(2+x)(3+x)}+\cdots$$
$$=\frac{e^{-1}}{1+x}+\frac{2^2}{2!}\,\frac{e^{-2}}{2+x}+\frac{3^3}{3!}\,\frac{e^{-3}}{3+x}+\cdots.$$

Putting $x=0$ and multiplying across by e yields

$$\sum_{n=1}^{\infty}\frac{(n/e)^{n-1}}{n!}=e.$$

II. *Solution by Robert Frucht, Technical University Santa Maria, Valparaiso, Chile.* The function $f(x)$ defined by the power series

$$f(x)=x+\frac{2x^2}{2!}+\frac{3^2x^3}{3!}+\frac{4^3x^4}{4!}+\cdots, \qquad |x|\leqq 1/e,$$

is the solution of the functional equation $f(x)=xe^{f(x)}$. (See Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. I, Aufgabe III, 209, pp. 125 and 301. See also problem 4418 [1952, 335], formula 9.) Hence, for $x=1/e$,

$$f(1/e)=\frac{1}{e}\sum_{n=1}^{\infty}\frac{(n/e)^{n-1}}{n!}$$

is the solution of the equation

$$f(1/e)=(1/e)e^{f(1/e)}.$$

It follows immediately that $f(1/e)=1$, whence the value of the given series is e.

III. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* If we expand e^{az} in powers of w where $z=we^{bz}$ we obtain

$$\frac{e^{az}-1}{aw}=\sum_{n=1}^{\infty}\frac{[(a+nb)w]^{n-1}}{n!}.$$

(See Bromwich, *Infinite Series*, p. 160, ex. 4.) Letting $z=1$, and then $w=e^{-b}$, there results

$$\frac{e^{a}-1}{ae^{-b}}=\sum_{n=1}^{\infty}\left[\frac{a+nb}{e^{b}}\right]^{n-1}\frac{1}{n!}.$$

The proposed problem is a special case of this where we let $b=1$ and $a\to 0$.

Also solved by J. S. Frame, Samuel Goldberg, V. C. Harris, Vern Hoggatt, H. Kaufman and J. Shipman, N. S. Mendelsohn, M. R. Spiegel, O. E. Stanaitis, P. Ungar, Chih-yi Wang, J. G. Wendel, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of The Association.

General College Mathematics. By W. L. Ayres, Cleota G. Fry, and H. F. S. Jonah. New York, McGraw-Hill Book Co., 1952. xii+283 pp. $3.75.

This well-executed book has been designed for use in a "terminal" one-year course in mathematics by students who have studied plane geometry and one year of high school algebra. It is pitched at exactly the proper level of difficulty for such a course while many comparable books are either too light or too difficult when there is much of a break from the traditional algebra, trigonometry, analytic geometry organization. Here the topics are introduced gently, but in very short order the reader encounters solid material.

In this volume the style is bright and clear, with a fine sprinkling of historical allusions throughout. These and the references to practical applications are skillfully woven into the text, unlike books where such matter, if present at all, has been relegated to a footnote status. The problems, too, have frequently a colorful and practical flavor. They should give the opportunity for enough practice to fix the ideas without too much pure drill work. The figures are good and typographical flaws are nearly non-existent.

Although one can criticize adversely almost any particular selection of subject matter for a book of this kind, these authors seem to have made a reasonable selection. Portions of the work usually done in college algebra, in mathematics of finance, and in elementary statistics courses are included, along with some fifty pages (with the explicit purpose of giving a broad view of mathematics) which have to do with numbers, topology, and logical principles. The range of ideas for a one-year course would be tremendous except that each discussion is carried to just about the right point to give some insight without going too far. Thus, in classroom use parts of chapters need not be omitted nor time wasted in trying to answer questions motivated by the text alone but actually beyond the student.

No real unity in organization from beginning to end is found. Instead of this, there are several rather unrelated sections. In particular, the last chapter is on the art of reasoning. The application of this chapter in the rest of the book is not emphasized for the student, although a good job of guiding the reader through the reasoning process is consistently done. To be sure, some flexibility for one who wishes to reorder or to omit sections is perhaps gained from the lack of unity.

A few sections, some of which should be mentioned, did not please the reviewer. For example, in introducing frequency distributions the discussion of locating class marks and end points of class intervals seems confused. A literal reader, encountering the subject for the first time, might get several erroneous

ideas. Once over this hurdle, the rest of the material from statistics, including a fair amount of sampling theory, is accurate and understandable. Then, in fifty pages of trigonometry there are thirty-five before an angle is properly defined or before the trigonometric functions of angles other than positive acute angles are given. This seems to be too much even if one prefers to consider acute angles first rather than as special cases in a more general treatment. At another point, the statement is made about ratios that "the quantities compared in this way must represent measures of the same kind and dimensions." The idea of a rate could have been at least mentioned here. Moreover, in the next paragraph the ratio of a number of degrees Fahrenheit to a number of degrees Centigrade is considered allowable while the ratio of a number of feet to a number of yards is not.

This reviewer deplores the tendency in some mathematics texts toward cuteness. Although the tendency is not painfully apparent here and the humor in the problems is mildly pleasant, some of it is present. Puzzle problems throughout the text interrupt the continuity slightly but probably add to the general attractiveness. The several very good features, and especially the accuracy with which it seems to fit the purpose for which it was designed, should make this book very useful, particularly in view of the scarcity of texts exactly appropriate for college terminal mathematics courses.

B. H. GERE
Hamilton College

Tensor Analysis: Theory and Applications. By I. S. Sokolnikoff. New York. John Wiley and Sons, Inc. 1951. 9+335 pages. $6.00.

The latest in a sequence of texts, some written in collaboration with another author, this book extends the list in several senses, including quality. For this one, Professor Sokolnikoff is both sole author and editor.

The material presented was developed out of courses of lectures given at various universities in the United States over a period of years. The target is the graduate student interested in applied mathematics or in the applications of mathematics. Thanks to the organization of the material and to the thorough explanations given or referred to, such students should be able to use the book with considerable success.

The book may be thought of as consisting of three parts, *viz.*, (1) introduction, (2) development of the theory, and (3) applications. Although these are not disjoint, they are nearly enough so to permit separate discussion. The introduction is contained in chapter one titled "Linear Vector Spaces. Matrices," and is intended to fill undesirable near-voids frequently found in the mathematical experience of the student to whom the book is addressed. This material is convenient but not indispensable, since it appears in many other places in expanded form. Chapter two titled "Tensor Theory" makes up the second part of the book. The presentation proceeds in orderly fashion from a discussion of coordinate transformations to the idea of admissible transformations and induced

transformations, and finally to tensors. Intuition, even of the tutored variety, will be of little help to the student here, but the orderliness of the development will simplify the work of the instructor basing his course on this book. Included are discussions of the Ricci and Einstein tensors, the Bianchi identities, generalized Kronecker deltas, and the Riemann tensor.

Part three requires all of the last two thirds of the book, pagewise, for its developments. The problem of selecting materials to illustrate possible applications of tensor theory, never solved in a manner satisfactory to all concerned, is approached here by working first with geometry. Since the theoretical developments in part two make only cursory reference to geometry, the treatment is rather interesting. The topics in differential geometry usually covered in texts on vector analysis are treated in a way which should help students to gain a comfortable inner understanding of the theory. For this reason, later topics in geometry and other applications should be rather simpler than would otherwise be the case. The remainder of part three is contained in the last three chapters covering analytical mechanics, relativistic mechanics, and mechanics of continuous media.

The book would seem to be rather easier to read than some recent similar works such as *Tensor Calculus* by Synge and Schild. The choice of materials for the applications is different too, in the latter, where the authors have omitted relativity theory, but have included a chapter on non-Riemannian spaces.

As might be expected in any material whose symbolism involves numerous subscripts and superscripts (even compounded), there are many typographical errors here. However, most of them are of a nature such that they will not lead to confusion. Indeed, it might be feasible for the instructor to make proofreading assignments.

The book is a valuable addition to the fund of texts on the subject, falling as it does in the space between terminal sections of many vector analysis texts and more advanced works on tensors.

WADE ELLIS
Oberlin College

NEW BOOKS RECEIVED

Essential Business Mathematics. Second Edition. By L. R. Snyder. New York, McGraw-Hill Book Company, 1953. x+421 pages. $4.50.

College Algebra and Plane Trigonometry. By J. H. Zant. Boston, Ginn and Company, 1953. x+337+82 pages. $4.00.

Harald Bohr, Collected Mathematical Works. Published by the Danish Mathematical Society with the support of the Rask-Orsted Foundation and the Carlsberg Foundation. Edited by Erling Folner and Borge Jessen. 3 Volumes of about 1000 pages each. Danish Mathematical Society (Institute of Mathematics, University of Copenhagen, Blegdamsvej 15, Copenhagen), 1952. Bound in buckram, total price 110 Danish Crowns (or 15.90 U.S. $ post free).

The Higher Arithmetic. An Introduction to the Theory of Numbers. By H.

Davenport. London, England, Hutchinson's University Library, 1952. (Obtainable through Longmans, Green and Co., Inc., 55 Fifth Avenue, New York 3, New York). 172 pages. $2.25.

Algebra for College Students. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1953. ix+394 pages. $3.75.

Table of Bessel-Clifford Functions of Orders Zero and One. National Bureau of Standards, Applied Mathematics Series 28. 1953. 72 pages. 45 cents. (Order from Government Printing Office, Washington 25, D.C.)

Introduction to Measure and Integration. By M. E. Munroe. Cambridge, Mass., Addison-Wesley Press. 1953. x+310 pages. $7.50.

The Learning of Mathematics, Its Theory and Practice. By National Council of Teachers of Mathematics. 1953. ix+355 pages. $4.00.

Tables of Chebyshev Polynomials $S_n(x)$ *and* $C_n(x)$, $n=2(1)12, x=0(.001)2$. National Bureau of Standards Applied Mathematics Series 9. 1953. 161 pages, $1.75. Government Printing Office, Washington, D. C.

Differential Equations. By A. L. Nelson, K. W. Folley and M. Coral. Boston, D. C. Heath and Company, 1952. x+299 pages. $3.75.

Introduction to the Theory of Functions of a Complex Variable. By W. J. Thron. New York, John Wiley and Sons, 1953. ix+230 pages. $6.50.

An Introduction to Statistics. By C. E. Clark. New York, John Wiley and Sons. 1953 x+266 pages. $4.25.

Fundamental Concepts of Geometry, Preliminary Edition. By B. E. Meserve. Cambridge, Mass., Addison-Wesley Publishing Co., 1953. 8+296 pages. $6.50.

Introduction to the Theory of Stochastic Processes Depending on a Continuous Parameter. Bureau of Standards Applied Mathematics Series 24. 45 pages, 30 cents. Government Printing Office, Washington 25, D. C.

Differential and Integral Calculus. By Philip Franklin. New York, McGraw-Hill Book Company, 1953. xi+641 pages. $6.00.

Stochastic Processes. By J. L. Doob. New York, John Wiley and Sons, Inc., vii+654 pages, 1953, $10.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB NEWS

(Clubs are invited to contribute news items)

The Mathematics Club of LaSalle College was installed on May 19, 1953 as

the Pennsylvania Beta Chapter of Kappa Mu Epsilon.

The Sacramento State College Math Club recently originated two interesting projects, the publication of a mathematics journal and the sponsoring of an annual contest. The publication, *Math Echoes*, is devoted to the interests of students and teachers of high school mathematics. A few copies of the initial number are available for readers of this MONTHLY who request them. Write to Professor Louis C. Graue, Sacramento State College, Sacramento 19, California. The first annual mathematics contest for high school seniors in the area served by the College was held during the week of May 18–22, 1953. The contest consisted of a forty-minute multiple-choice examination over elementary algebra and geometry, the students taking the examination at the participating high schools. First and second place winners were awarded scholarships at Sacramento State College. Certificates of honorable mention were presented to those who placed high.

The ninth biennial convention of Kappa Mu Epsilon was held on May 17–18, 1953 at the Michigan Education Association Camp, St. Mary's Lake, Michigan. The hosts were Michigan Alpha of Albion College, Michigan Beta of Central Michigan College, and Michigan Gamma of Wayne University. More than 150 delegates attended. Featured on the two-day program were eight student papers:

Napierian logarithms, by Gary Brown, Drake University

Some mathematical principles underlying visual art, by Bernadine Law, Mount St. Scholastica College

Certain geometrical properties of the tesseract, by Henry Beersman, Southwest Missouri State College

A device for drawing the locus of points, by Raoul Pettai, Colorado A & M College

Mathematics—pure or otherwise, by Nancy Marsh, Washburn Municipal University, Topeka

The line, circle, and parabola using complex numbers, by Joe E. Ballard, North Texas State College

Solving a differential equation on a differential analyzer, by Joseph Weizenbaum, Wayne University

A 1953 bridge problem, by Nancy Hartup, Baldwin-Wallace College.

Prizes for the best papers were awarded as follows: first place, Raoul Pettai; second place, Bernadine Law; third place, Joseph Weizenbaum. National officers elected for the next biennium are: president, Charles B. Tucker; vice-president, C. C. Richtmeyer; secretary, E. Marie Hove; treasurer, M. Leslie Madison; historian, Laura Z. Greene.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INTERNATIONAL CONGRESS OF MATHEMATICIANS

The International Congress of Mathematicians of 1954 will be held in Amsterdam, The Netherlands, on September 2–9 under the auspices of Het Wiskundig Genootschap (The Mathematical Society of the Netherlands). Those who wish to attend the Congress are requested to send name, qualifications including title and degrees, and address to the secretariat, 2d Boerhaavestraat 49, Amsterdam, The Netherlands, as soon as possible.

FREQUENCY RESPONSE SYMPOSIUM

A Frequency Response Symposium will be held on December 1–2, 1953 at the Hotel Statler, New York City, in conjunction with the annual meeting of the American Society of Mechanical Engineers. Details can be obtained from the Symposium Chairman, Dr. Rufus Oldenburger, chief mathematician, Woodward Governor Company, Rockford, Illinois.

DANNIE HEINEMAN PRIZE

The Board of Directors of the Heineman Foundation for Research, Educational, Charitable and Scientific Purposes, Incorporated, announce the establishment of a prize to be known as the "Dannie Heineman Prize." This prize of $5,000 is to be awarded every three years to the author of an outstanding book or manuscript in the mathematical or physical sciences. The object of the prize is to encourage the writing of books on a high scientific level which have merits of exposition and which are likely to facilitate access to important fields of research. Submissions for the next award shall be made not later than December 31, 1955. Further information may be obtained from the Secretary of the Foundation, 50 Broadway, New York 4, New York.

PERSONAL ITEMS

Professor S. W. McCuskey of Case Institute of Technology represented the Association at the inauguration of President G. B. Earnest of Fenn College on May 9, 1953.

Mr. S. T. Paine was the representative of the Association at the inauguration of President M. A. Love of San Diego State College on May 10, 1953.

Professor Haim Reingold, Illinois Institute of Technology, represented the Association at the Special Convocation of the University of Chicago which was held on May 8, 1953 in celebration of the Fiftieth Anniversary of the Law School.

Sister Marie Grace, vice-president of Barry College, was appointed to repre-

sent the Association at the inauguration of President J. F. W. Pearson of the University of Miami on May 7, 1953.

Assistant Professor Mary Sunseri of Stanford University was the representative of the Association at the inauguration of President J. T. Wahlquist of San Jose State College on May 1, 1953.

Professor L. V. Ahlfors of Harvard University and Professor Einar Hille, Yale University, have been elected to membership in the National Academy of Sciences.

Mr. T. A. Brown, formerly a student at the State University of Iowa, has been awarded a Rhodes scholarship for the academic year 1953–54.

Professor Arthur Erdélyi has been elected a foreign member of the Academy of Sciences of Turin, Italy.

Dean Gillie A. Larew of Randolph-Macon Woman's College has been awarded the honorary degree of Doctor of Humane Letters by the College. Dean Larew has retired after an association of more than fifty years with the College as student, faculty member, and dean.

Professor K. O. May of Carleton College has been awarded a Ford Foundation Fellowship for the academic year 1953–54 for "the preparation of materials for the mathematical training of social scientists."

Professor C. F. Stephens, head of the Department of Mathematics of Morgan State College, has been awarded a Ford Foundation Fellowship for the year 1953–54 and will study at the Institute for Advanced Study.

Denison University announces the following: Associate Professor Marion D. Wetzel has been granted sabbatical leave for the first semester of the academic year 1953–54; Assistant Professor A. C. Ladner of the Departments of Mathematics and Engineering Science has retired.

At De Paul University: The Department of Mathematics is inaugurating a new program of studies in mathematics intermediate to the Master's Degree and the Ph.D. Degree; students completing this program of study will be granted a certificate designating them as Mathematics Specialists; requirements include forty-eight semester hours of graduate work in mathematics, a reading knowledge of French, German, or Italian, a comprehensive written examination, and the completion of an oral examination.

Kent State University reports the following: Professor L. L. Lowenstein has asked to be relieved of his duties as Head of the Department of Mathematics but is continuing to serve as Professor of Mathematics; Professor L. E. Bush of the College of St. Thomas has been appointed Head of the Department of Mathematics.

Lawrence Institute of Technology announces the following: Professor H. W. Nace has been appointed Head of the Department of Mathematics; Dr. D. E. Coffey, formerly head of the Department of Mathematics of Saline High School, Michigan, has been appointed to an assistant professorship.

University of Illinois announces: Professor W. G. Madow has received a grant from the Fund for the Advancement of Education for the year 1953–54 and is spending the year at Princeton University; Assistant Professor Franz Hohn has been granted leave for 1953–54 and has accepted a one-year visiting research appointment with the Bell Telephone Laboratories; a sabbatical leave has been granted to Associate Professor Wilfrid Wilson for the first semester of 1953–54; Professor Abraham Taub has been granted a sabbatical leave for the second semester of 1953–54 and will visit England during this time.

Dr. A. F. Bausch of the University of Chicago has been promoted to an assistant professorship.

Dr. Benjamin Bernholtz of the University of Connecticut has accepted a position as a mathematician with the Research Division of the Hydro-Electric Power Commission of Ontario, Toronto, Canada.

Mr. A. J. Bosgang has accepted a position as a systems analyst with the Computing Systems Engineering Section, R.C.A. Victor Division, Camden, New Jersey.

Associate Professor Angeline J. Brandt of Wheaton College has been promoted to a professorship.

Mr. R. G. Brown, formerly at the Willow Run Research Center, Michigan, is now a member of the Operations Research Group, Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Mr. Ermon H. Bryant, previously head of the Department of Mathematics of Northwest Mississippi Junior College, has accepted a position as a mathematician with the Air Force Armament Center, Eglin Air Force Base, Florida.

Mr. H. P. Carter has returned to his position as an instructor at David Lipscomb College after a period of military service.

Professor J. A. Daum of Texas Agricultural and Mechanical College is on leave of absence during the year 1953–54 and is engaged in a research problem for the Research Foundation of the College.

Dr. Robert Davies, formerly an associate engineer with the Rand Corporation, Santa Monica, California, has a position as Senior Research Engineer at the General Motors Research Laboratories, Detroit, Michigan.

Dr. A. H. Diamond, previously chief of the Mathematics Branch, United States Army Office of Ordnance Research, has been appointed to a professorship at Stevens Institute of Technology.

Dr. R. L. Ely, who has been in the United States Air Force, has a position as an engineer at the Applied Physics Laboratory, Johns Hopkins University.

Dr. G. B. Findley of the University of Florida has a position as a physicist with the United States Navy Mine Countermeasures Station, Panama City, Florida.

Assistant Professor W. C. G. Fraser of Dartmouth College has accepted a position with the McLennan Laboratories, University of Toronto.

Dr. D. L. Fuller of the General Aniline and Film Corporation has accepted

a position as Technical Director with the Fortier Plant of the American Cyanamid Company, New Orleans, Louisiana.

Associate Professor B. H. Gere of Hamilton College has been promoted to a professorship and has been named to the Samuel F. Pratt Chair in Mathematics.

Assistant Professor A. M. Gleason of Harvard University has been promoted to an associate professorship.

Assistant Professor Samuel Goldberg of Lehigh University has been appointed to an assistant professorship at Oberlin College.

Associate Dean P. H. Graham of the Washington Square College of Arts and Sciences, New York University, has retired.

Professor H. E. H. Greenleaf of DePauw University has been promoted to the position of Head of the Department of Mathematics.

Graduate Assistant L. A. Guest of the University of Oklahoma has been appointed Manager of the Davenport Telephone Company, Oklahoma.

Dr. D. C. Harkin, who has been at the Naval Research Laboratory, Washington, D. C., is now a consultant with the Department of Defense, Washington, D. C.

Mr. E. E. Heckroth, previously a computer at the Bell Aircraft Corporation, Niagara Falls, New York, has been promoted to the position of Dynamics Engineer.

Associate Professor Banesh Hoffmann of Queens College has been promoted to a professorship.

Mr. O. H. Hoke of the Naval Air Test Center, Patuxent River, Maryland, has accepted a position as a mathematician at the White Sands Proving Ground, New Mexico.

Assistant Professor Eugene Isaacson of New York University has been promoted to an associate professorship.

Mr. I. H. Kral has a position as a technical assistant with Sandia Corporation, Albuquerque, New Mexico.

Mr. C. Y. Lee of the University of Washington is now a member of the technical staff of the Bell Telephone Laboratories, New York City.

Dr. R. A. Leibler, formerly with the Sandia Corporation, has accepted a position as a mathematician with the Department of Defense, Washington, D. C.

Associate Professor C. C. Lin of Massachusetts Institute of Technology has been promoted to a professorship.

Dr. Harold Luxenberg, previously a member of the technical staff of Hughes Research and Development Laboratories, Culver City, California, is now a consulting engineer with the Eckert-Mauchly Division of Remington Rand, Incorporated, Philadelphia, Pennsylvania.

Mr. D. L. McIntosh, who has been teaching at South Denver High School, Colorado, is now at Magdalen College, Oxford University, England.

Professor L. L. Merrill, director of graduate studies, Clarkson College of

Technology, has been appointed Professor and Head of the Department of Mathematics, Stevens Institute of Technology.

Assistant Professor E. D. Nering of the University of Minnesota is on leave of absence and has a position at the Goodyear Aircraft Corporation, Litchfield Park, Arizona.

Mr. C. S. Ogilvy, formerly a part-time instructor at Syracuse University, has been appointed to an instructorship at Hamilton College.

Mr. B. E. Owens of Yale University has a position as an assistant project engineer with the Sperry Gyroscope Company, Great Neck, New York.

Dr. C. L. Perry, previously at the Oak Ridge National Laboratory, has been appointed to an assistant professorship at the United States Naval Postgraduate School.

Professor G. C. Priester, head of the Department of Mathematics and Mechanics of the University of Minnesota, has retired.

Associate Professor A. L. Putnam has succeeded Professor E. P. Northrop as Chairman of the Mathematics Staff in the College of the University of Chicago.

Assistant Professor R. A. Roberts of West Virginia University has been appointed to an assistant professorship at the University of Miami.

Mr. Wellington Rounds, Jr., formerly a student at Rutgers University, is now at the Air Force Cambridge Research Center, Cambridge, Massachusetts.

Assistant Professor R. D. Schafer of the University of Pennsylvania has been appointed Professor of Mathematics and Head of the Department of Mathematics at the University of Connecticut.

Dr. W. J. Schart has been promoted to a position of Design Specialist with the Electronics Laboratory, Convair, San Diego.

Associate Professor Henry Scheffé of the Department of Mathematical Statistics of Columbia University has been appointed Professor of Mathematics and Assistant Director of the Statistical Laboratory of the University of California, Berkeley.

Professor W. H. Sellers of Alderson-Broaddus College has accepted a position as a flutter and vibration technologist with Fairchild Aircraft Division, Hagerstown, Maryland.

Mr. Judson C. Smith of the University of Washington has a position as a mathematician at the Naval Ordnance Test Station, China Lake, California.

Mr. B. R. Snyder has received an appointment as Assistant to the Director of Benefits and Pensions Business, New England Mutual Life Insurance Company, Boston, Massachusetts.

Mr. R. B. Stiles, who has been at White Sands Proving Ground, Las Cruces, New Mexico, has returned to his position as Assistant Professor of Mathematics at Vanderbilt University.

Assistant Professor C. T. Taam of the University of Missouri has been appointed to an assistant professorship at Catholic University of America.

Dr. Fumio Yagi, formerly of the University of Washington, is at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Professor Emeritus D. R. Curtiss of Northwestern University died on April 28, 1953 in Redlands, California. He was a charter member of the Association.

Professor Emeritus Theodore Lindquist of Michigan State Normal College died on March 26, 1953. He was a charter member of the Association.

Associate Professor W. L. Pickard of Northern Illinois State Teachers College died on December 27, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1953 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	E. B. Miller, Illinois College
Iowa	D. L. Holl, Iowa State College
Louisiana—Mississippi	F. A. Rickey, Louisiana State University
Maryland—Dist. of Col.—Virginia	S. B. Jackson, University of Maryland
Michigan	P. S. Jones, University of Michigan
Minnesota	K. O. May, Carleton College
Philadelphia	C. O. Oakley, Haverford College
Southern California	C. W. Trigg, Los Angeles City College
Texas	E. R. Heineman, Texas Technological College

As is usually the case, many of these elections have resulted in close contests. As a result a high percentage of votes has been cast in many of the Sections. The highest percentage of votes cast was 59% in the Iowa Section.

The majority of the members of the Board of Governors are elected by the Sections of the Association. In this way active participation in the affairs of the Association is afforded to representatives who reflect the wide geographical distribution of the membership of the Association.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1953 COMBINED MEMBERSHIP LIST

The Mathematical Association of America is issuing each year a Combined Membership List jointly with the American Mathematical Society. The 1953 edition will be sent free of charge to all members of the Association about December 1.

The office of the Association should be notified promptly of all changes in rank, position, and address which have not previously been reported. The final date for receipt of changes is October 15. Any errors in the 1952 Membership List should also be reported before October 15.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 150 persons have been elected to membership by the Board of Governors on applications duly certified.

REZZUK ADEM, Student, Detroit Institute of Technology.

A. R. AMIR-MOEZ, M.A. (U.C.L.A.) Grad. Student, University of California at Los Angeles.

A. C. ANDERSEN, M.A. (Columbia) Teacher, Ben Franklin High School, Rochester, N. Y.

M. I. AUERBACH, M.A. (Southern California) Instr., East Los Angeles Junior College.

H. W. BAKER, Student, Nebraska Wesleyan University.

J. E. BARKER, M.A. (California) Mathematician, United States Naval Proving Ground, Dahlgren, Va.

R. E. BARLOW, Student, Knox College.

G. D. BATH, B.A. (Illinois) Editor, *The Peoria Star*, Peoria, Ill.

NORMAN BAUMAN, Student, Harvard University.

L. D. BAUMERT, B.A. (U.C.L.A.) Electronics Repair Officer, United States Navy.

JACK BELZER, B.S. in E.E. (Cooper Union) Director, Computation Laboratory, Battelle Memorial Institute, Columbus, Ohio.

M. P. BERRI, B.S. (Rockhurst) Teaching Fellow, University of Notre Dame.

S. I. BIRNBAUM, B.A. (N.Y.U.) Part-time Instr., Polytechnic Institute of Brooklyn.

B. L. BLANKENSHIP, M.A. (Texas) Teacher, San Saba High School, Texas.

LEE BROGAN, B.S. (Denver) Staff Research Mathematician, Denver Research Institute of the University of Denver.

A. L. BROWN, M.S. (Howard) Mathematician, United States Navy Hydrographic Office, Washington, D. C.

MRS. MIRIAM H. BROWN, Student, University of Buffalo.

T. A. BROWN, Student, State University of Iowa.

R. J. BUEHLER, Ph.D. (Wisconsin) Staff Member, Sandia Corporation, Albuquerque, N. M.

G. H. BUTCHER, Ph.D. (Pennsylvania) Asst. Professor, Howard University.

J. E. CAGLE, Student, University of Mississippi

PETER CALBERSON, Diamond Cutter, J. Kinstlinger, New York City.

R. P. C. CALDWELL, Student, Harvard University.

BUCHANAN CARGAL, M.A. (North Texas State) Instr., Iowa State College.

R. B. CASE, Student, University of Buffalo.

WILLIAM CAUGHLIN, M.A. (U.C.L.A.) Instr., East Los Angeles Junior College.

ROBERT CHAFE, M.A. (Columbia) Teacher, Sydney Academy, Nova Scotia, Canada.

G. E. COLLINS, M.S. (Iowa) Grad. Student, State University of Iowa.

ELLEN CORREL, B.S. (Rutgers) Grad. Teaching Assistant, Purdue University.

MARGARET J. COTTER, Student, New Jersey State Teachers College, Upper Montclair, N. J.

ETHELDRA F. COX, Student, University of Mississippi.

D. F. CRILEY, B.A. (Occidental) Grad. Student, University of California.

C. M. CRULL, Student, University of Mississippi.

H. F. DAVIS, II, B.S. (M.I.T.) Part-time Instr., Massachusetts Institute of Technology.

SOL DAVIS, B.S. (Brooklyn) Grad. Student, Carnegie Institute of Technology.

BETTY C. DETWILER, Student, Kentucky Wesleyan College.

S. F. DICE, Litt.M. (Pittsburgh) Instr., University of Detroit.

N. J. DIVINSKY, Ph.D. (Chicago) Asst. Professor, University of Manitoba.

W. C. DIXON, B.A. (Connecticut) Mathematician, United States Naval Proving Ground, Dahlgren, Va.

L. K. DURST, Ph.D. (C.I.T.) Instr., Rice Institute.

M. H. M. ESSER, Ph.D. (Northwestern) Asst. Professor, Georgia Institute of Technology.

S. G. EVANS, President, Newfane Lumber and Manufacturing Company, Lockport, N. Y.

HARRY FAIRBAIRN, B.A. (New Brunswick) 2/Lt., Lord Strathcona's Horse, Calgary, Alberta, Canada.

R. M. FARMER, B.S. (Florida State) Mathematician, Eglin Air Force Base, Fla.

C. M. FAST, Research Analyst, Bendix Aviation, Los Angeles, Calif.

W. H. FAUX, Student, American University.

R. B. FREUND, M.A. (Southern California) Instr., East Los Angeles Junior College.

J. R. GILLIS, B.S. (Aquinas) Teaching Fellow, University of Detroit.

SEYMOUR GINSBURG, Ph.D. (Michigan) Asst. Professor, University of Miami.

OSCAR GOLDMAN, Ph.D. (Princeton) Asst. Professor, Brandeis University.

WILLIAM GONZALEZ, 156 East 118th Street, New York City.

I. F. GOODMAN, B.S. (Wayne) Mathematics Engineer, Chrysler Corporation, Detroit, Mich.; Grad. Student, Wayne University.

R. M. GORDON, M.A. (Yale) Mathematician, United States Naval Ordnance Test Station, China Lake, Calif.

E. E. GRACE, B.S. (North Carolina) Grad. Student, University of North Carolina.

NORMAN GREENSPAN, M.E.E. (Poly. Inst. of Brooklyn) Instr., Polytechnic Institute of Brooklyn.

JOSEPH GRUENEBAUM, Student, Brooklyn College.

SIMON GRUENZWEIG, Ph.D. (Pittsburgh) Professor and Chairman of Department, Philander Smith College.

MILTON HANDEL, B.A. (U.C.L.A.) Mathematician, United States Naval Air Missile Test Center, Pt. Mugu, Calif.

Virginia S. Hanly, P. O. Box 64, Frankfort, Ky.

ANNA M. HARRIS, M.A. (Virginia) Asst. Professor, North Carolina State College.

REID HAYWOOD, M.A. (Appalachian S.T.C.) Mathematician, United States Naval Proving Ground, Dahlgren, Va.

G. F. HEWITT, M.A. (Wisconsin) Chairman of Department, Von Steuben High School, Chicago, Ill.

J. L. HILDEBRAND, M.A. (North Texas State) Asst. Professor, Western Washington College of Education.

S. B. HOBBS, M.S. (New Hampshire) Instr., University of New Hampshire.

MARK HOLZMAN, B.A. (N.Y.U.); B.S. (Valparaiso Tech. Inst.) Research Engineer, Western Geophysical Company, Los Angeles, Calif.

W. J. HORN, M.S. (Michigan) Engineer, Western Electric Company, Kearney, N. J.

MARGARET A. IMLAY, Student, American University.

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J. R. JACKSON, M.S. (New Zealand) Assistant Master, Greymouth Technical High School, New Zealand.

BERNARD JACOBSON, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College.

D. G. JOHNSON, Student, Albion College.

R. D. JOHNSON, JR., B.A. (Dartmouth) Grad. Student, University of Virginia.

C. L. JOHNSTON, M.A. (Southern California) Instr., East Los Angeles Junior College.

L. G. KELLY, M.S. (Minnesota) Asso. Professor, Clemson Agricultural College.

C. A. KENT, Student, University of Mississippi.

HEWITT KENYON, B.S. (California) Instr., University of Rochester.

G. J. KLEINHESSELINK, M.A. (Northwestern) Fellow, Northwestern University.

C. G. KOCH, Student, Marquette University.

D. B. KUNSTLER, Student, Brown University.

P. M. LANDRY, B.S. (Southeastern Louisiana) Associate Mathematician, Eglin Air Force Base, Fla.

P. E. LAUDERBACH, Research Chemist, Columbia-Southern Chemical Corporation, Barberton, Ohio; Student, University of Akron.

S. J. LAWRENCE, M.A. (N.Y.U.) Instr., New York State Institute of Applied Arts and Sciences, Brooklyn.

H. D. LECHNER, Student, University of Kansas.

EUGENE LEVINE, B.A. (Brooklyn) Pvt., United States Army.

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H. H. MARTENS, Tester, Consolidated Edison Company of New York, New York City; Student, Cooper Union.

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T. P. MULHERN, Student, Fordham University.

ELIZABETH M. MULLER, M.A. (Columbia) Mathematician, Remington Rand Laboratory, South Norwalk, Conn.

E. D. NERING, Ph.D. (Princeton) Asst. Professor, University of Minnesota.

P. J. NIKOLAI, Student, College of St. Thomas.

A. B. J. NOVIKOFF, B.A. (Brown) Research Associate, Institute for Cooperative Research, Johns Hopkins University.

R. R. O'BRIEN, Student, Massachusetts Institute of Technology.

MRS. INA S. OLSON, M.Ed. (Montana S. U.) Head of Department, Custer County High School and Junior College, Miles City, Mont.

W. E. PACE, M.A. (Virginia) Instr., University of Virginia.

E. M. PEASE, Ph.D. (M.I.T.) Professor and Head of Department, University of Rhode Island.

H. P. PETERSON, Student, Ohio University.

E. I. PINA, Student, University of Massachusetts.

L. E. PUMMILL, Ph.D. (George Peabody) Head of Department, State College, Springfield, Mo.

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A. R. SCHMIDT, M.S. (Purdue) Grad. Teaching Assistant, Purdue University.

J. R. SCHUE, Student, Macalester College.

R. E. SCHWARTZ, M.S. (Chicago) Instr., Ripon College.

V. L. SHAPIRO, Ph.D. (Chicago) Instr., Rutgers University.

BETTIE L. SHIPMAN, M.S. (Texas Tech.) Teaching Fellow, Texas Technological College.

SISTER M. MIRIAM, Ph.D. (Detroit) Teacher, St. Clare Academy, Sylvania, Ohio.

SISTER MARY CHARLOTTE (Fowler) Ph.D. (Catholic) Chairman of Department, Nazareth College, Louisville, Ky.

SISTER MARY CORONA (Deditz), M.S. (Marquette) Head of Department, Cardinal Stritch College, Milwaukee, Wisc.

MICHAEL SKALSKYJ, Dr.rer.nat. (Göttingen) Instr., Xavier University.

J. C. SMITH, B.S. (U. of Washington) Mathematician, United States Naval Ordnance Test Station, China Lake, Calif.

D. B. SPARKS, B.A. (Texas A. & M.) Computer, Sun Oil Company, Beaumont, Tex.

SARAH J. SPRINGER, B.A. (Vassar) Teacher, Moberly Junior College, Mo.

BROTHER ANDREW STEPHEN, B.A. (Manhattan) Head of Department, De La-Salle Collegiate, Detroit, Mich.

H. M. THAXTON, Ph.D. (Wisconsin) Lecturer-Engineer, City College of the City of New York.

J. D. THOMAS, Student, University of Oklahoma.

J. G. THOMPSON, Student, Yale University.

S. L. THORNDIKE, Ph.D. (California) Asso. Professor and Head of Department, College of Emporia.

A. J. TINGLEY, Ph.D. (Minnesota) Instr., University of Nebraska.

C. J. TIRMAN, JR., Student, University of California at Los Angeles.

EUGENIA TRAPP, Student, Wesleyan College, Macon, Ga.

W. R. TROTT, B.A. (Mississippi) Grad. Fellow, University of Mississippi.

D. A. TRUMPLER, M.A. (British Columbia) Grad. Assistant, Massachusetts Institute of Technology.

J. D. TUPAC, M.S. (Minnesota) Mathematician, United States Naval Air Missile Test Center, Pt. Mugu, Calif.

C. J. VANDERLIN, JR., M.S. (Wisconsin) Teaching Assistant, University of Wisconsin.

A. W. WALKER, M.A. (Toronto) Lecturer, University of Toronto.

E. F. WARD, M.A. (George Peabody) Instr., Tennessee Polytechnic Institute.

PATRICIA A. WARD, Student, Texas Christian University.

W. D. WARD, Student, University of Michigan.

JACK WARGA, Ph.D. (N.Y.U.) Principal Engineer, Republic Aviation Corporation, Farmingdale, N. Y.

C. E. WATTS, Mus.B. (Drury) Grad. Student, University of California.

H. E. WEBB, M.S.E. (Michigan) Assistant Engineer, Hazeltine Corporation, Little Neck, N. Y.

J. E. WEIDLICH, JR., M.S. (Stanford) Grad. Student, University of California.

JOHN WOODWARD, B.S. (Presbyterian) Grad. Student and Research Assistant, University of Georgia.

A. W. YONDA, B.S. (Alabama) Grad. Assistant, University of Alabama.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The fifteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at San Francisco State College, San Francisco, on January 31, 1953. Professor R. M. Robinson, Chairman of the Section, presided at the morning session, and Professor Roy Dubisch, Vice-Chairman, presided at the afternoon session.

There were ninety-one persons present, including the following fifty members of the Association:

H. L. Alder, H. M. Bacon, T. J. Bass, Alice K. Bell, M. T. Bird, W. E. Bleick, G. E. Breden, A. C. Burdette, L. P. Burton, R. C. Campbell, Randolph Church, Roy Dubisch, Hazel E. Eggett, G. C. Evans, F. D. Faulkner, Ruth A. Fish, Harley Flanders, R. R. Fossum, C. M. Fulton, L. C. Graue, W. R. Hanson, J. G. Herriot, Marjorie L. Hoffman, H. F. Hunter, V. F. Ivanoff, Verne James, T. A. Jeeves, Walter Jennings, Helene G. Kusick, R. M. Lakness, B. J. Lockhart, A. R. Lovaglia, Sophia L. McDonald, F. R. Morris, W. H. Myers, C. D. Olds, H. A. Osborn, H. J. Osner, C. L. Perry, J. P. Pierce, F. M. Pulliam, C. H. Rawlins, R. M. Robinson, E. B. Roessler, Abraham Seidenberg, Mary V. Sunseri, Irving Sussman, Gabor Szegö, C. C. Torrance, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Roy Dubisch, Fresno State College; Vice-Chairman, Professor J. G. Herriot, Stanford University; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

A resolution concerning the preparation of teachers, recently adopted by the Joint Committee on Mathematical Education of the Northern California Sec-

tion and the Southern California Section of the Mathematical Association of America, was proposed by Professor Sophia L. McDonald. The resolution was adopted by unanimous vote. The resolution is as follows:

"*Whereas* it has been found that approximately two-thirds of people teaching arithmetic in the ninth and tenth grades have neither teaching majors nor minors in mathematics, and

Whereas the number of pupils reaching this level and requiring instruction in arithmetic will certainly increase markedly in the immediate future, and

Whereas an alarming shortage of candidates for the General Secondary Credential with teaching majors or minors in mathematics even now exists,

Therefore be it resolved that teacher training institutions be urged to require *all candidates* for the General Secondary Credential, as well as all candidates for the Elementary Credential, to demonstrate competence in arithmetic and in the teaching of arithmetic, and that such institutions provide suitable courses wherein such competence can be acquired."

By invitation of the Section, Professor P. L. Morton, Professor of Electrical Engineering, University of California, Berkeley, delivered an address at the morning session entitled "Electronic Digital Computers."

The following papers were presented:

1. *Some remarks on The Faculty Fellowship Program of the Fund for the Advancement of Education*, by Professor Roy Dubisch, Fresno State College.

Professor Dubisch first described the purpose of the Faculty Fellowships of the Fund for the Advancement of Education, and then reported on the work he did during 1951–52 as a Fellow of this foundation. This work consisted of the following: (1) a survey of the use of mathematics in physics, engineering, *etc*; (2) a survey and compilation of suggestions for the improvement of mathematics teaching; and (3) observation of the mathematics program at the College of the University of Chicago.

2. *On the geometry of approach for maxima of functions of two variables*, by Professor G. C. Evans, University of California, Berkeley.

For a function of two variables Professor Evans gives an elementary topological discussion of the neighborhood of a critical point, in the case where the sections corresponding to the axes are concave downward. If a maximum then exists, a certain step-by-step process converges, but otherwise diverges. This process is analogous to a competitive process in economics, (*Stability of limited competition and cooperation*, with K. May, Reports of a Mathematical Colloquium, Notre Dame University, ser. II, issue 1, 1939, pp. 3–15), but applied to a trust where total profit is to be maximized. Neither the equilibrium position to be arrived at nor the cost function of one's associate need be known.

3. *Perfect numbers*, by Professor R. M. Robinson, University of California, Berkeley.

A perfect number is one which is the sum of its proper divisors. Euclid showed that the number $2^{n-1}(2^n-1)$ is perfect if 2^n-1 is prime. Euler showed that all even perfect numbers are of Euclid's form; no odd perfect numbers are known. The problem of finding even perfect numbers is thus reduced to that of finding primes of the form 2^n-1, known as Mersenne primes. There are now seventeen Mersenne primes known, corresponding to $n=2$, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107,

127, 521, 607, 1279, 2203, and 2281; this list is believed to be complete for $n<2300$. The first twelve were discovered by 1914; the twelfth $2^{127}-1$, was in fact identified by Lucas as early as 1876. Using Lucas' test, suitably coded by the speaker for the SWAC (the National Bureau of Standards' Western Automatic Computer), the next five Mersenne primes were found in 1952. They are at present the five largest primes of any sort which are known; the largest $2^{2281}-1$, has 687 digits.

4. *Modification of the Graeffe method for obtaining the roots of a polynomial*, by Professor F. M. Pulliam, United States Naval Postgraduate School.

Various simplifications of the Graeffe method are given. Obtaining the moduli of the roots of the equations $F(z)=0$, $F(z'+a)=0$, and $F(z''+b)=0$ by the Graeffe algorithm, we can get the roots of the equation $F(z)=0$ as the points lying on three families of circles. Or, knowing the modulus R we obtain e so that the remainder $p(e)z+q(e)$ when $F(z)$ is divided by z^2+ez+R^2 is zero. This can be done by trial or by obtaining the zeros of the *GCD* of $p(e)$ and $q(e)$. Using another modulus we then work on the quotient, *etc.*

5. *Inequalities in the theory of partitions*, by Professor H. L. Alder, University of California, Davis.

In the *Bull. Amer. Math. Soc.*, vol. 54, 1948, pp. 712–722, the author proved that certain generalizations of the Rogers-Ramanujan identities and a theorem of I. Schur cannot exist. That other generalizations of Schur's theorem are impossible is furnished by the following theorem: The number of partitions of n into parts differing by at least d, each part being greater than or equal to $m(m\leqq d)$, where a part divisible by d differs from the nearest other part by at least $e(e>d)$, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever if $d\geqq 3$, unless $d=3$, $m=1$.

6. *Useful exercises connected with the ellipse and hyperbola*, by Professor A. R. Williams, University of California, Berkeley.

Problems and exercises often arise which present no marked difficulty but offer something helpful and stimulating to the student. Four such topics are mentioned: (1) the relations connecting the angular magnitudes which are called in astronomy the true anomaly, the eccentric anomaly, and the mean anomaly; (2) the modification in a Mercator map when the earth is regarded as an ellipsoid of revolution instead of a sphere; (3) the region of the plane from which four real normals can be drawn to an ellipse and the locus of points at which these four are harmonic; (4) the representation of certain integrals by arc lengths on a hyperbola, done by MacLaurin, 1740.

C. D. OLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 13–14, 1953 at Alabama Polytechnic Institute, Auburn, Alabama. Professor H. K. Fulmer, Chairman of the Section, and Professor W. V. Parker, Vice-Chairman, presided over the general sessions; Professors H. K. Fulmer, W. V. Parker and F. A. Lewis presided over subsections.

There were about two hundred in attendance including the following one hundred eight members of the Association:

P. L. Armstrong, E. A. Bailey, W. S. Beckwith, R. C. Boles, J. P. Brewster, S. K. Bright, G. M. Brown, Helen E. Brown, B. F. Bryant, C. M. Callahan, E. A. Cameron, Virginia Carlock, T. C. Carson, Ella F. Casey, N. A. Childress, Howard Chitwood, R. S. Christian, B. G. Clark, A. C. Cohen, Jr., J. B. Coleman, C. L. Cope, R. W. Cohan, J. C. Currie, A. Lois Dicks, L. A. Dye,

J. C. Eaves, D. O. Ellis, F. A. Ficken, Ben Fitzpatrick, Jr., D. J. R. Foulis, H. K. Fulmer, Wallace Givens, D. B. Goodner, S. T. Gormsen, H. C. Griffith, E. H. Hadlock, B. F. Hadnot, S. W. Hahn, O. G. Harrold, Jr., E. A. Hedberg, Marguerite Z. Hedberg, G. W. Hess, A. T. Hind, Jr., J. H. Hoelzer, J. H. Hornback, C. W. Huff, G. B. Huff, S. L. Hull, P. M. Hummel, Ernest Ikenberry, S. L. Jamison, Ayrlene M. Jones, H. W. Kelley, F. W. Kokomoor, J. W. LaGrone, C. G. Latimer, F. A. Lewis, J. L. Locker, G. H. Lundberg, Nathaniel Macon, J. M. Marr, Elna B. McBride, E. P. Miles, Jr., D. C. Moore, R. H. Moorman, J. C. Morelock, J. H. Moss, W. B. Moye, Frances A. Norton, J. D. Novak, W. V. Parker, W. D. Peeples, Jr., C. G. Phipps, L. T. Ratner, J. M. Robertson, A. J. Robinson, H. A. Robinson, F. Virginia Rohde, W. C. Royster, W. A. Rutledge, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, A. R. Sloan, W. S. Snyder, D. E. South, E. L. Stanley, L. W. Stark, W. L. Strother, William Charles Taylor, S. L. Thompson, W. G. Thornton, H. S. Thurston, J. H. Wahab, J. G. Wall, Susie L. Ward, R. E. Wheeler, M. C. Wicht, Ernest Williams, W. L. G. Williams, R. A. Willoughby, R. L. Wilson, F. J. Witt, Elizabeth S. Wolf, Herbert Wolf, G. N. Wollan, J. W. Young, B. K. Youse.

The following officers were elected for the coming year: Chairman, Professor W. V. Parker, Alabama Polytechnic Institute; Vice Chairman, Professor W. L. Williams, University of South Carolina; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *The Pythagorean theorem: proof number one thousand, with a new slant*, by Professor J. C. Eaves, Alabama Polytechnic Institute.

A square is constructed on the hypotenuse of a right triangle so that the triangle is part of the square. This square is cut, pivot-connected, and then "unwound" to form two squares, one of which could be constructed on the long leg of the triangle, the other on the short leg of the triangle. This demonstrates the obviousness of the theorem in the general case. Formal proof is given by using, as a heuristic basis, the "unwound" figure as if it were constructed in a superposed position on the original figure.

2. *What the colleges and universities are doing about the poorly prepared student in mathematics*, by Professor W. L. Williams, University of South Carolina.

Based on a questionnaire sent to over five hundred colleges, it was revealed that a large percentage of these institutions have lost confidence in high school mathematics units as indicators of students' preparation and are requiring mathematics placement tests for entrance. Because of poor preparation a great many colleges are giving remedial courses in algebra and geometry.

3. *A note on radical equations*, by Professor B. G. Clark, Vanderbilt University.

The equation containing three radicals was discussed as a generalization of that in this MONTHLY, vol. 59, 1952, p. 320. Conditions for one, two, or no solutions were derived, and it was pointed out how, in several ways, radical equations with suitable roots may be constructed.

4. *An approach to solving equations in X*, by Professors J. H. Wahab and R. A. Willoughby, Georgia Institute of Technology.

The basic idea is that each equation in X "partitions" a given set of numbers into two subsets, one of which consists of the numbers "satisfying" the equation. A logical analysis based on this partitioning facilitates the understanding of solving equations, introduces set-theoretical concepts in a natural way, and lays a foundation for more advanced mathematical reasoning.

5. *On introducing arguments into freshman mathematics*, by Professor G. B. Huff, University of Georgia

It is generally agreed that beginning students need to do mathematical reasoning in their beginning courses. Professor Huff gave particular types of problems which lend themselves to accurate arguments and showed how these arguments might be made.

6. *Concerning freshman mathematics texts*, by Dr. J. B. Coleman, Orlando, Florida

Most freshman algebras are illogical in that they use terms technically without any introductory discussion of them. In chapters on finance many definitions of rate are not accurate and the term rate is used indiscriminately as a rate or a ratio.

7. *A need for more symbolism in college teaching*, by Mr. C. W. Huff, University of Georgia.

The statements of theorems and definitions in elementary texts are often obscure or not precise. Mr. Huff suggested that this situation can be improved by using a standard symbolism.

8. *Some problems of teaching mathematics at the United States Navy Airman School*, by Professor J. T. Moore, Georgia Institute of Technology, introduced by the Secretary.

This paper suggests unusual ways to meet the great disparity in preparation of these students.

9. *A note on integration of products of powers of the secant and cosecant when one of the powers is even and the other odd*, by Professor J. W. Young, University of Florida.

In this MONTHLY, vol. 48, 1914, p. 544, W. W. Burton and W. G. Miller gave a trigonometric method for evaluating $\int \csc^n x\,dx$, $\int \sec^n x\,dx$, and $\int \sec^m x \csc^n x\,dx$ for m, n both even or both odd. The present paper removes this restriction and shows that for m even, n odd

$$\int \sec^m x \csc^n x\,dx = \int \left(\sum_{j=0}^{\frac{1}{2}(n-1)} a_j \csc^{2j+1} x + \sum_{j=0}^{\frac{1}{2}(m-2)} b_j \sec^{2j} x \cdot \sec x \tan x \right) dx,$$

and for m odd, n even

$$= \int \left(\sum_{j=0}^{\frac{1}{2}(n-1)} c_j \sec^{2j+1} x + \sum_{j=0}^{\frac{1}{2}(n-2)} d_j \csc^{2j} x \cdot \csc x \cot x \right) dx,$$

to each of which the above methods are applicable.

10. *Some relationships between size of school and academic achievement of high school seniors in Florida*, by Professor R. C. Boles, Florence State Teachers College.

An analysis of 109,110 test scores of 22,057 Florida high school seniors in 1950 and 1951 indicated little association between school size and achievement in mathematics of seniors with similar intelligence scores. The interdependence of school size and achievement in English, social studies, natural sciences, and mathematics was tested using 221 contingency tables in conjunction with the chi-square statistic.

11. *College arithmetic; a summer program for teachers*, by Professor M. C. Wicht, North Georgia College

The author discusses one solution of the problem of course content for secondary school teach-

ers who take a terminal mathematics course to satisfy certification requirements. With the chief objective of getting at the roots of difficulties that arise in collegiate mathematics, the course content, texts available and the author's experiences in giving such a course are discussed.

12. *Linear measure versus path-length*, by Professor O. G. Harrold, Jr., University of Tennessee.

In this paper the notion of linear measure for a curve is compared with the path-length of a given representation of the curve. The word curve may be interpreted in either the sense of a Jordan continuous curve (=Peano continuum) or a curve in the Menger-Urysohn sense. Several inequalities are given comparing the corresponding numbers, and the possibility of attaining certain lower bounds is discussed. A sharp inequality is given. A connection to certain topological problems is mentioned.

13. *Pseudo square roots and rational valued metrics*, by Professor E. B. Shanks, Vanderbilt University.

The pseudo square root based on b of any non-negative integer n is defined to be $\sqrt{n}\rceil = (b^2n+a^2)/2ab$ where a is the unique non-negative integer such that

$$a^2 \leqq b^2n \leqq a^2 + 2a$$

and b is any fixed positive integer. A proof is given that each pseudo square root of $(x_2-x_1)^2 + (y_2-y_1)^2$ is a metric assuming only rational values on the space of integral pairs (x, y) which assigns distances between pairs of points approximately the same as the Euclidean metric and that this approximation can be made as close as desired by choosing b large enough.

14. *Determinants of matrices with elements in a division ring*, by Professor Wallace Givens, University of Tennessee.

Let A have elements in a division ring K. One easily proves $A = \prod_a(l_n+u_av_a)$, where u_a is n by l and v_a is l by n. Since l_n+uv is similar to l_n+rs if and only if the scalars vu and sr are conjugate in the multiplicative group of $K-O$ and $(l_n+uv)^{-1}$ exists $(=l_n-u(l+vu)^{-1}v)$ only if $l+vu \neq 0$, define $\Delta(A) = \prod_a(l+v_au_a)$ for all factorizations of A. Then $\Delta(A)$ is 0 or a conjugate class in $K-O$ in agreement with Dieudonné (*Bull. Soc. Math. France* 71, 1943, pp. 27–45).

15. *Properly primitive ternary indefinite quadratic genera of more than one class*, by Professor E. H. Hadlock, University of Florida.

Let each of the odd factors Ω' and Δ' of the invariants Ω and Δ respectively of a form f consist of a square factor and a common factor which is the product of distinct primes. Then it is shown that genera of at least two classes exist containing properly primitive indefinite forms with determinants of odd or even values and with properly primitive reciprocals.

16. *The Abel-Riesz method of summation*, by Dr. B. F. Hadnot, Florida State University.

The author did not intend that this method be a product method. Rather, if $\sum A_ne^{-\sigma\lambda_n}(\sigma>0)$ is summable (R, λ_n, K) to $C_\lambda^K(\sigma)$ and $C_\lambda^K(\sigma)\to C$ as $\sigma\to 0^+$, then $\sum A_n$ is said to be summable (A, R, λ_n, K) (Abel-Riesz) to C. For this method, arithmetic and abelian theorems have been established.

17. *The coefficient problem for P-valent functions*, by Professor W. C. Royster, Alabama Polytechnic Institute.

It is a well-known conjecture that if $f(z) = \sum_{n=1}^{\infty} a_nz^n$ is regular and univalent in $|z|<1$ that $|a_n| \leqq n|a_1|$. A similar conjecture, which reduces to the one above when $p=1$, has been made for

p-valent functions. Certain subclasses of p-valent functions are discussed for which this conjecture holds.

18. *On the isomorphism of LF (2, n) with a collineation group of degree n*, by Professor F. A. Lewis, University of Alabama.

If $S=(\delta_{j,k}E^{j-1})$, $T=(\delta_{j+1,k})$, where $E=e^{2\pi i/n}$, represent collineations on n variables, then the collineation group of this paper is the largest collineation group on n variables, with coefficients and variables in the field of complex numbers, under which $\{S, T\}$ of order n^2 is invariant.

19. *Map of a system of algebraic curves invariant under a cyclic involution of period eleven*, by Professor S. T. Gormsen, University of Florida.

In this paper a system of algebraic curves, invariant under an I_{11} is mapped onto a surface F in space S_7. It is found that the curves have a five-tuple point on F, with tangent lines determined by the intersection of a fifth order cone and a hyperplane.

20. *Convergence over sets of planes*, by Professor L. T. Ratner, Vanderbilt University.

A class of iterative processes for the "solution" of systems of linear equations (*i.e.*, yielding the usual solution for consistent systems of equations) is investigated. Necessary and sufficient conditions for convergence of the processes are established. Particular interest lies in the case in which the number of equations exceeds the number of unknowns.

21. *A note on a cyclic involution of period five*, by Professor N. A. Childress, University of Florida.

The image of the planar cyclic involution of period five can be represented as a surface of order five in S_4. The plane is in (5, 1) correspondence with this surface. This note shows the existence of a rational surface of order twenty-five in S_8 which is in (5, 1) correspondence with the image surface.

22. *Cut points and cut sets*, by Professor W. L. Strother, University of Miami.

R. L. Moore's theorem on the existence of points which do not cut a space X is extended by A. D. Wallace to a result concerning points which do not cut certain subsets of X. This paper extends Wallace's theorem to a theorem concerning sets which do not cut certain other sets.

23. *Set-theoretic description of normal topologies*, by Professor D. O. Ellis, University of Florida.

Professor Ellis demonstrated the theorem: If S is a normal T_1 space, there is a biuniform mapping K of S onto a totally unordered subset of a set algebra so that if x_n is a net (cf. J. L. Kelley, *Convergence in topology*, Duke Math. Journal, vol. 17, 1950, pp. 277–283) on some directed set to S, then x_n converges to x in S if and only if $K(x)\subset\lim_n K(x_n)$.

24. *Some observations on undergraduate mathematics in American colleges and universities*, by Professor E. A. Cameron, University of North Carolina.

This paper has been published in this MONTHLY, vol. 60, 1953, pp. 151–155.

25. *Motives and trends in mathematics*, by Professor H. K. Fulmer, Georgia Institute of Technology.

This paper presents the thesis that in the field of science generally there exists a basic conflict between the profit or utilitarian motive and the creative motive. As a consequence it is questioned

whether our vast research bureaus will yield a return.

26. *The reductio ad absurdum method*, by Professor C. G. Phipps, University of Florida.

The "reductio ad absurdum" method is not a method of proof; it is a method by which consistent assumptions are selected from a list of assumptions necessarily containing all possible assumptions. The method is particularly useful where only one possible assumption is consistent with the previous results. If all the listed assumptions are found to be inconsistent, this fact indicates one of two things: either the list is incomplete or else the previous assumptions are themselves inconsistent.

27. *Multivalued functions*, by Mr. D. J. R. Foulis, University of Miami.

It is shown that the multivalued function, as ordinarily defined (See W. Simonsen, *Acta Mathematica*, vol. 81, pp. 291–297) is nothing but a (dyadic) relation, and that all of the theorems of (dyadic) relation theory are applicable to multivalued function theory. Simonsen's principal theorems are proved in what seems to be a more satisfying manner by making use of some of the properties of the ancestral relation as defined in *Principia Mathematica*, vol. I, by Russell and Whitehead.

28. *When is Banach space a Hilbert space?* by Professor F. A. Ficken, University of Tennessee.

Any one of several simple geometric restrictions on a complex linear normed complete (*i.e.*, Banach) space B imply that there can be defined on $B \times B$ a bilinear scalar product (x, y) with $\overline{(x, y)} = (y, x)$ and $\|x\|^2 = (x, x)$, thus making B a Hilbert space. Results obtained by several investigators are summarized.

29. *A note on the algebraic determination of the p-th roots of unity*, by Professor J. C. Morelock, Alabama Polytechnic Institute.

This paper calls attention to the relationship between the "Pascal triangle" and the coefficients of the completed equation obtained in the algebraic solution of $x^p = 1$. Not only is the work simplified in completing the reduced equation but several interesting identities in the binomial coefficients arise out of these considerations.

30. *On the relation between Euler and Cesàro methods of summability for double series*, by Professor G. N. Wollan, Memphis State College.

This paper showed that, as in the case of simple series, if a double series is summable by both C_1 and E_1 methods then the two sums agree, but that neither method includes the other.

31. *The summation of certain types of series*, by Professor F. Virginia Rohde, University of Florida.

Given a function $\phi(x)$ whose finite integral is $f(x)$ with interval of differencing h. For k any positive integer an expression for $\sum_{x=1}^{n} \phi(x)$ is found in terms of $f(x)$ taken between suitable limits.

32. *Some remarks about operations research*, by Professor A. C. Cohen, Jr., University of Georgia.

This paper is concerned primarily with the role of Operations Research in the United States Armed Forces. The solution of a bombing accuracy problem serves to illustrate a phase of this activity in the Air Force.

33. *A neglected method for the resolution of polynomial equations*, by Dr. W. C.

Taylor, Jr., Air Proving Ground, Eglin Air Force Base.

Dr. Taylor discussed the method for resolution of polynomial equations proposed by Daniel Bernoulli in 1728. For a great many problems this little known theory presents the quickest means of obtaining an answer to any required precision. In particular, the cubic can always be solved. When modern machine methods of calculation are used, any cubic equation can be completely resolved in a fraction of the time used by other methods.

34. *Self-dual postulates for projective geometry*, by Professor M. H. M. Esser, Georgia Institute of Technology, introduced by the Secretary.

In projective three-dimensional geometry, any theorem remains true when we replace everywhere the words points by planes and planes by points. To bring out this symmetry, or "dualism" between points and planes, we show that points and planes can be defined by postulates which already exhibit this dualism. A generalization to n dimensions is possible. (*Duke Mathematical Journal*, vol. 18, 1951, pp. 475–479).

35. *Reflection in a circle*, by Professor L. A. Dye, The Citadel.

The direction of a ray through a fixed point A within a circle which passes through another fixed point B after n reflections from the circle is calculated. Cases which are constructible by use of a straight edge and compass are given especial attention.

36. *Abnormal semantic matrices*, by Professor E. P. Miles, Jr., Alabama Polytechnic Institute.

Semantic (having meaningful rows and columns) matrices which remain semantic under an elementary transformation (inversion, reversion or reflection) are called abnormal (ipso-abnormal, if invariant). Examples of these and other semantic matrices are given, but to avoid censorship or libel suits no perverted or derogatory matrices are given.

37. *On terminating solutions of certain linear differential equations with three term recurrence relations*, by Professor R. W. Cowan, University of Florida.

The technique of solving the differential equation by Frobenius' method leads to a three term recurrence relation. The requirement that at least one solution should terminate imposes several conditions on the coefficients of the equation. Three illustrative examples are included, two of which involve a parameter.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-third regular meeting of the Southern California Section of the Mathematical Association of America was held at Los Angeles City College on March 14, 1953. Mr. C. W. Trigg, Chairman of the Section, presided.

The attendance was one hundred fifteen, including the following seventy-nine members of the Association:

L. J. Adams, O. W. Albert, Frances C. Amemiya, Florence R. Anderson, Norman Anning, T. M. Apostol, L. A. Aroian, Leon Bankoff, J. L. Barnes, Mabel S. Barnes, Lulu Bechtolsheim, William Beck, May M. Beenken, Clifford Bell, Paul Brock, R. E. Bruce, Herbert Busemann, P. H. Daus, D. O. Davidson, R. P. Dilworth, D. C. Duncan, D. L. Elliott, Arthur Erdelyi, G. E. Forsythe, F. W. Gibson, W. H. Glenn, B. K. Gold, J. W. Green, C. J. A. Halberg, H. J. Hamilton, V. C. Harris, A. F. Herbst, R. B. Herrera, M. R. Hestenes, P. G. Hoel, J. M. Howell, P. B. Johnson, H. J. Jones, G. R. Kaelin, Rosella Kanarik, P. J. Kelly, L. J. Lander, L. C. Lay, D. H. Lehmer, J. W. Lindsay, Fred Marer, R. B. McMurdo, W. H. Mead, Jr., E. E. Moots, J. S. Morison,

F. R. Morris, J. B. Nelson, W. A. Nielsen, L. J. Paige, W. D. Paxton, Jr., W. T. Puckett, R. M. Redheffer, E. C. Rex, L. G. Riggs, J. M. Robb, Samuel Skolnik, Alton H. Smith, James C. Smith, Ernst Snapper, T. H. Southard, Maria A. Steinberg, R. A. Struble, J. D. Swift, T. E. Sydnor, A. E. Taylor, Elmer Tolsted, C. W. Trigg, S. E. Urner, F. A. Valentine, Morgan Ward, P. A. White, J. V. Whittaker, A. D. Wirshup, E. M. Zaustinsky.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor A. E. Taylor, University of California at Los Angeles; Vice-Chairman, Professor E. C. Rex, George Pepperdine College; Secretary-Treasurer, Professor P. H. Daus, University of California at Los Angeles; Chairman of Program Committee, Professor D. H. Hyers, University of Southern California.

The following program was presented:

1. *Finite partially ordered sets*, by Professor R. P. Dilworth, California Institute of Technology.

This paper contains a brief account of the structure of finite partially ordered sets with special emphasis upon homomorphisms.

2. *The differential analyzer in elementary instruction*, by Professor R. M Redheffer, University of California at Los Angeles.

The mechanical analogue-type differential analyzer is described, with occasion taken by the way to illustrate various items occurring in the undergraduate curriculum. The discussion is wholly expository.

3. *Some problems in convex bodies*, by Professor Herbert Busemann, University of Southern California.

The theory of Finsler spaces has raised problems concerning convex bodies of a novel type. They concern the sections of a convex body with center by planes through the center, in contrast to the standard theory of convex bodies which mainly deal with, or may be reduced to, the study of parallel plane sections. The new problems are all understandable to anyone and look therefore quite simple; never-the-less most of them are involved and those which have been solved proved non-trivial.

4. *Current application of mathematics in the aircraft industry*, by Professor J. L. Barnes, North American Aviation and the University of California at Los Angeles.

Current applications of arithmetic, algebra, analysis and geometry are discussed with emphases on automatic solutions methods and equipment. Topics of present interest to aircraft-industry applied mathematicians, such as block-diagram topology, prediction and information theory, and the synthesis of systems represented by non-linear integro-differential equations are briefly touched upon. As an illustration of the role played by machine methods, the present method for designing autopilots is outlined.

5. *On the number of elements of given period in finite symmetric groups*, by Mr. R. B. Herrera, Los Angeles City College.

Let $N(k; r)$ represent the number of elements of period r in the symmetric group of degree k. The following recurrence formula is proved:

$$N(k;r) = \sum_{s|r} \frac{(k-1)!}{(k-s)!} \sum_{b|r} N(k-s;b), \quad \text{with} \quad [s,b] = r.$$

In the present note, the author gives a new and shorter proof of a result first presented in 1947.

6. *Some sophisms in mathematics*, by Professor Norman Anning, University of Michigan.

The speaker presented a few simple lessons, learned from experience with students, teachers, textbooks, and the angle-trisecting human race.

7. *Notes on the teaching of vector analysis*, by Professor E. C. Rex, George Pepperdine College.

Prerequisites of vector analysis at Pepperdine College are the calculus and solid analytic geometry. The students who have this background also have had general physics. Vectors are represented by underlined letters. Unit vectors, with certain exceptions, also have overbars. Vector sums and products are treated thoroughly. Position vectors and their applications are then shown. Comparisons of vector analysis and complex numbers are made. Covariant and contravariant components are graphically illustrated. The parts played by dyadics in rotating, distorting, and projecting vectors are emphasized. Differentiation and integration of vectors introduce the gradient, divergence, curl, potential, and theorems generally taught elsewhere.

8. *New methods for solving an old problem*, by Professor M. R. Hestenes, University of California at Los Angeles.

This paper is concerned with an apparently new method of solving a system of simultaneous linear equations. The method is described in a paper by Stiefel and Hestenes entitled *The methods of conjugate gradients for solving linear systems* published in the *Journal of Research*, Department of Commerce, vol. 49, 1952, pp. 409–436.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The fortieth annual meeting of the Iowa Section was held jointly with the sixty-fifth annual meeting of the Iowa Academy of Science and the twentieth convention of the Junior Academy of Science of Iowa at Cornell College, Mount Vernon, Iowa, on April 17–18, 1953. The Chairman, Professor L. A. Knowler, the Vice-Chairman, Professor J. O. Chellevold, and the Secretary, Professor Fred Robertson, presided in turn.

The total of seventy-five persons, including the following thirty-six members of the Association registered their attendance:

A. H. Blue, I. H. Brune, Buchanan Cargal, J. O. Chellevold, E. W. Chittenden, G. E. Collins, N. B. Conkwright, A. T. Craig, W. M. Davis, B. E. Gillam, H. E. Goheen, Cornelius Gouwens, F. S. Harper, B. H. Henry, R. V. Hogg, D. L. Holl, A. A. Karwath, L. A. Knowler, O. C. Kreider, C. H. Lindahl, W. D. Lindstrom, C. G. Maple, R. B. McClenon, B. E. Meserve, H. T. Muhly, H. V. Price, Fred Robertson, Hazel M. Rothlisberger, Augusta L. Schurrer, M. F. Smiley, F. M. Stein, H. P. Thielman, H. C. Trimble, Bernard Vinograde, L. A. Ware, Roscoe Woods.

Officers elected for the year are: Chairman, Professor J. O. Chellevold, Wartburg College; Vice-Chairman, Professor M. F. Smiley, State University of Iowa; and Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented:

1. *Generalizations of continuity*, by Mr. Buchanan Cargal, Iowa State College.

The author has submitted the paper for publication in the *Proceedings of the Iowa Academy of Science.*

2. *Logarithms to base i*, by Professor B. H. Henry, Parsons College.

Using the formula $(a+bi)^{c+di}=r^c e^{-d(2\pi n+\theta)}\cdot[\cos d(2\pi m+\ln r)+i \sin d(2\pi m+\ln r)]\cdot[\cos c(2\pi n+\theta)+i\sin(2\pi n+\theta)]$, where $r=\sqrt{a^2+b^2}$, $\theta=\arctan b/a$ in radians, $e=$natural logarithm base, $\ln r=\log_e r$, m and n are any positive integers, the author showed that

$$\log_i n=\frac{-2i\ln n}{\pi} \tag{1}$$

where n is a positive real number. Since $\log_i(+1)=4k$, $\log_i(-1)=4k+2$, $\log_i(+i)=4k+1$, and $\log_i(-i)=4k+3$, where $k=0, 1, 2, 3, \cdots$, it follows that the logarithm to the base i of any real or pure imaginary number may be expressed as a complex number.

The relation (1) also was shown to hold when n is complex. By use of the relation $\ln(\cos x+i\sin x)=ix$, it is possible to find the natural logarithm of a complex number by putting the number in its trigonometric form. Knowing $\ln n$ is sufficient to compute $\log_i n$.

3. *Chain molecules and probability*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

A starch molecule is a chain of units; hydrolysis breaks the bonds joining the units, always leaving two or three still joined. We seek the probable distribution of unbroken bonds, once the action is completed. Sillen and Myrback have solved the problem on the hypothesis of initial chains of infinite length. The present paper gives the solution for initial chains of any finite length; for ten or more units, it does not differ appreciably from the former Swedish one.

4. *A theorem in differential geometry*, by Mr. W. D. Lindstrom, State University of Iowa.

Let $\Sigma=F\langle y\rangle$ be a differential extension field of a differential field (d-field) F of characteristic zero. The least integrally closed d-subring of Σ containing $R=F\{y\}$ is called the *d-integral closure* of R in Σ. This ring need not coincide with the integral closure of R. If Σ is of degree of transcendency one over F, and F is maximally algebraic in Σ, it is proved that the d-integral closure, $\bar{R}$, of R is a finite integral domain. From this result it follows that, if F contains nonconstants, there is an element z in $\bar{R}$ such that $\Sigma=F\langle z\rangle$ and the integral closure of $F[z]$ is a d-subring. Then $F\{z\}=F[z, z_1, \cdots, z_r]$ for some r.

5. *Graphical solutions of all optimal strategies in a finite game*, by Professor Bernard Vinograde, Iowa State College.

For finite games a geometric method of solution which emphasizes the display of all optimal strategies is illustrated for several games. Briefly, the process is based on the following steps to be carried out for each player: (a) Solve a system of linear equalities, (b) Plot level lines of a support function.

6. *Algebra and trigonometry for the gifted student*, by Professor Fred Robertson, Iowa State College.

The author discussed the methods used to segregate and accelerate students at the Iowa State College who showed a very high level of competence at the time of admission.

7. *A remark on commutative arithmetic*, by Professor H. T. Muhly, State

University of Iowa.

Let the integral ideals A of a commutative integral domain R be mapped by a function f onto a partially ordered semi-group Γ, in such a way that: (1) $f(AB)=f(A)+f(B)$, (2) $f(A)\geqq f(B)$, when $A\leqq B$, (3) if $f(A)\leqq f(A_\beta)$ for all ideals of a set $\{A_\beta\}$, then $f(A)\leqq f(A_\beta)$. For any A let $\{A_\beta\}$ be the set of ideals such that $f(A)=f(A_\beta)$, and let $A'=\sum A_\beta$. Then $A'\in\{A_\beta\}$, and $A'\geqq A_\beta$ for all β. The mapping $A\to A'$ is a prime operation on the integral ideals in the sense of Krull and Prufer (W. Krull, *Beitrage zur Arithmetic (I)*, Math. Zeit., vol. 41, 1936) if and only if the projection of f on the principal ideals of R defines a valuation of the quotient field of R in the sense of Fuchs (L. Fuchs, *Generalization of the valuation theory*, Duke Math. Jour., vol. 18, 1951) for which R is the valuation ring.

8. *Some comments concerning the present opportunities for mathematicians*, by Professor L. A. Knowler, State University of Iowa.

A report was given concerning the needs for persons with considerable mathematical training. It was pointed out that positions requiring mathematical training in fields other than teaching have become more numerous in recent years and that the expanded influence of science on our daily pursuits seems to indicate that this trend will continue in the future. Among the most important of the fields indicated were: Statistics (including Operations Research, Biostatistics, and Statistical Quality Control); Applied Mathematics such as Mechanics, Electrodynamics, and Aerodynamics; Insurance; and Astronomy. The demand for additional training in Pure Mathematics in each of these fields, as well as in teaching, was stressed.

The increase in the level of salaries for mathematicians was pointed out. It was urged that administrators and executives in school systems, governmental agencies, and industry recognize this increase as being necessary to secure and to retain the type of person needed with such training.

9. *Navigation instruction of naval officer candidates*, by Professor J. O. Chellevold, Wartburg College.

10. *The path and orbit of the South Dakota detonating meteor of September 27, 1952*, by Professor C. C. Wylie and Mr. G. E. Collins, State University of Iowa, presented by Mr. Collins.

11. *A remark on an efficient statistic*, by Professor A. T. Craig and Professor R. V. Hogg, State University of Iowa, presented by Professor Hogg.

Let $f(x;\theta)$ denote the one parameter probability density, or the probability, of a random variable x according as x is a variable of the continuous or discrete type. Concerning the parameter θ it is proved: In a regular case of estimation, necessary and sufficient conditions that θ admit, in the sense of Cramér, an efficient statistic $\hat{\theta}=\hat{\theta}(x_1, x_2, \cdots, x_n)$, where $x_1, \cdots, x_n$ is a random sample of n values of x, are: (1) that $f(x;\theta)=\exp[g(\theta)K(x)+S(x)+q(\theta)]$ and (2) that $q'(\theta)/g'(\theta)=b-c\theta$, b and c constants with $c\neq 0$. Moreover, the efficient statistic is $\hat{\theta}=(1/nc)[K(x_i)+nb]$ and the (Cramér-Rao) minimum variance of $\hat{\theta}$ is $1/ncg'(\theta)$.

12. *On a cryptographic system*, by Professor H. E. Goheen, Iowa State College.

In Professor B. M. Stewart's new text on the theory of numbers is mentioned a cipher system based on the integers modulo 29 which, however, he says could probably be broken because the matrices involved would have to be small. This is true only if the enciphering is done by hand. If the calculations are done by a machine equipped to do arithmetic in a modular field, this drawback would not exist. The author describes a system for performing arithmetic modulo 31 and its mechanization and makes some comments on matrix computation with such a machine.

13. *Discussion of teaching problems*, by Professor W. M. Davis, Cornell College, Chairman; Mr. S. D. Nolte, Keokuk Junior High School; Mr. E. A. Groenendyk, Donnellson High School; and Miss Louise M. Gaekle, Muscatine Junior College.

Among the points considered by the panel were the following:

1. The uses of mathematics at various levels in our educational system and in everyday life.
2. The need for and the difficulties in teaching word or verbal problems.
3. The value of an adequate testing program in relation to grades and teaching.
4. Recognizing and meeting individual and class differences.
5. Influencing student's attitudes toward preparation of lessons.
6. Preparing students in small high school classes for participation in the larger college classes which may be encountered later.
7. Self-evaluation of the teacher and the importance of the teacher in the learning situation.

14. *Topology for secondary schools*, by Professor B. E. Meserve, University of Illinois. (By invitation).

A few elementary but fundamental topological concepts are considered in an effort to show that topology has significance and meaning for mathematics teachers and for pupils in secondary schools. Continuity, invariance under geometric transformations, simple open curves, simple closed curves, the Jordan curve theorem, topologically equivalent figures and surfaces, traversable graphs, the four color problem, and Möbius strips are discussed. Through the discussion there is an effort to use only concepts that can be understood by high school students.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on April 18, 1953 at Wayne University in Detroit in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. In the absence of the Chairman, Professor H. D. Larsen, Professor A. L. Nelson presided at the morning session, the luncheon and the business meeting. Professor L. E. Mehlenbacher presided at the afternoon session.

A total of eighty-one persons attended the meetings, including the following forty-seven members of the Association:

K. J. Arnold, J. W. Baldwin, R. C. F. Bartels, F. A. Beeler, M. Isobel Blyth, W. M. Borgman, R. G. Brown, Y. W. Chen, R. V. Churchill, C. J. Coe, S. D. Conte, Benjamin Epstein, H. G. Falahee, K. W. Folley, J. S. Frame, Caspar Goffman, E. W. Goings, Betty Grossman, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, L. S. Johnston, P. S. Jones, A. E. Lampen, K. B. Leisenring, Saunders MacLane, M. T. MacNeil, E. D. McCarthy, L. E. Mehlenbacher, H. W. Nace, A. L. Nelson, R. S. Pate, G. A. Paxson, Gertrude V. Pratt, G. Y. Rainich, P. H. Raker, M. O. Reade, B. V. Ritchie, L. J. Rouse, Sister Mary Paula, J. G. Sowul, H. E. Stelson, B. M. Stewart, D. T. Teodoro, Leonard Tornheim, J. B. Tysver, J. E. Vollmer.

At the business meeting the treasurer's report was read and approved. The nominating committee composed of Professors A. E. Lampen, R. V. Churchill, and C. M. Erikson, reported a slate of candidates from which the following officers were elected: Chairman, Professor J. S. Frame, Michigan State College; Secretary-Treasurer, Professor S. D. Conte, Wayne University.

A resolution thanking Professor P. S. Jones for his four years of service as Secretary-Treasurer was proposed by Professor R. V. Churchill and passed unanimously as was also a resolution proposed by Professor L. E. Mehlenbacher thanking our hosts, Wayne University, and in particular Professor S. D. Conte, chairman of local arrangements, and Professor A. L. Nelson, the presiding officer.

The following papers were presented at the morning and afternoon sessions:

1. *A theorem in function theory*, by Professor M. O. Reade, University of Michigan.

In a recent issue of the *Michigan Mathematical Journal* (vol. 1, 1952, pp. 169–185) W. Kaplan introduced a class of schlicht functions, defined in the unit disc, which he called "close-to-convex." This paper proved that the Bieberbach conjecture, concerning the coefficients of normalized schlicht functions, holds for a certain subclass of the "close-to-convex" functions; this subclass includes the well known "star" and "convex" functions.

2. *Distribution of lengths of chords of regular figures*, by Mr. R. G. Brown, University of Michigan, Willow Run Research Center.

Functional expressions, $P(c)$ and $p(c)$, were derived for the cumulative and probability density distributions, respectively, of the lengths, c, of chords of a square and of a circle. Three distinct definitions of the random variable are used for the square, and analogously, two for the the circle. For the square $P(c)$ has a jump at $c=1$, with $p(c)$ linear for $c<1$. For the circle of radius 1, $p(c)$ becomes infinite at $c=2$. The average length of chord is 0.68, 0.71, or 0.99 of the side of the square, depending on the definition. The average chord of the circle is either 1.57 or 1.27 times the radius, depending on the definition.

3. *Some remarks about the reliability of tests*, by Professor F. A. Beeler, Western Michigan College of Education.

The word "reliability" when applied to a test seems to have several meanings in the literature. In this paper two of these meanings were defined. It was shown that they are different, and some of the relationships between them were pointed out. Only the split-half and test-retest reliability were considered.

Definitions of what were called reliability 1 and reliability 2 are:

1. By *reliability* 1 of a test one means the degree to which
   (a) a subject tends to obtain the same scores on repeating the same test or on different but equivalent tests,
   (b) subjects with the same ability tend to obtain the same scores on a test.
2. By *reliability* 2 of a test one means the correlation of scores on the two halves of the same test or the correlation of scores made on two different but equivalent tests by the same group of subjects.

A test having perfect reliability 1 can have a reliability 2 ranging from zero to one. If a test has high reliability 2 then the group taking the test has different abilities, *i.e.* the group is heterogeneous. If a test has low reliability 2 then the group is fairly homogeneous, *i.e.* has about the same ability.

4. *The effective rate of interest*, by Professor H. E. Stelson, Michigan State College.

This paper showed how formulas for finding the rate of interest may be derived under the assumption that equations are formed with the factor $1+nr$ when payments are made before the focal date and the factor $1-nr$ when payments are made after the focal date. A series given in this

MONTHLY, (vol. 56, 1949), was inverted and continued fractions were used to obtain convergents as approximations to the interest rate. Error formulas were given.

5. *A report on the M.A.A.-N.C.T.M. Symposium on Teacher Training in Mathematics*, by Professor M. Isobel Blyth, Michigan State College and Professor Gertrude V. Pratt, Central Michigan College of Education.

This report was followed by discussion by members of the audience.

6. *European mathematical programs*, by Professor Saunders MacLane, University of Chicago. (By invitation).

In his invited hour address Professor MacLane discussed features of elementary and advanced mathematical programs at European mathematical centers as contrasted with American customs. In England, the Mathematical Association has had for 50 years a teaching committee, which has prepared influential reports on the teaching of Calculus, Trigonometry, Geometry, and other topics. On the Continent, courses have paid more attention than in the United States to the stimulus of the exceptional student; Calculus is taught in large lecture sessions; hard problems rather than routine ones are emphasized. In advanced mathematics, there is less emphasis on specialization in research and more on the interdependence of the various branches of active mathematics. This emphasis is especially striking in the work of Bourbaki.

7. *Some aspects of two-dimensional space-time*, by Professor K. B. Leisenring, University of Michigan.

Zero distances in the space-time metric occur only between points which are collinear with a fixed point at infinity, and hence may be regarded as parallel points, and the lines which are "self-perpendicular" are at an infinite angle from other lines, and thus are not metrically proper lines of the geometry.

Angles between two space-like lines are properly measured by positive imaginaries and those between time-lines by negative imaginaries (apart from sense), while the angle between perpendiculars is $\pi/2$. It follows that angles add to π about a point on one side of a line and the angle sum of a triangle is π. The straight line is not the shortest distance between its ends, but is the longest of properly comparable curves.

8. *Binomial and hyperbolic series coefficients modulo p^n*, by Professor J. S. Frame, Michigan State College.

We define the binomial coefficients $\binom{n}{k}$ and certain hyperbolic series coefficients a_k, b_k, and e^k by the series $(1+x)^n=\sum\binom{n}{k}x^k$, $2\tanh(x/2)=\sum_1^\infty a_k x^{2k-1}/(2k-1)!$, $(x/2)\coth(x/2)-1=\sum_1^\infty b_k x^{2k}/(2k)!$, $1-\operatorname{sech} x=\sum_1^\infty e_k x^{2k}/(2k)!$; remarking that $a_k=2(4^k-1)b_k/k$, $b_k=(-1)^{k-1}B_k$, $e_k=(-1)^{k-1}E_k$, $(e_0=0)$, where B_k are the Bernoulli numbers and E_k the Euler numbers. We consider an odd prime $p=2q+1$, and arbitrary integers m, $n>0$. Then since every term in the expansion of $[(1+x)^p-1-x^p]^n\cdot(1+x)^m$ is divisible by $(px)^n$ we obtain the following congruence among binomial coefficients:

$$\sum_{s,t}(-1)^s\binom{n}{s}\binom{s}{t}\binom{m+np-sp}{k-tp}\equiv 0 \pmod{p^n}.$$

The coefficients a_k and e_k are given recursively by formulas $\sum_s a_s\binom{2k}{2s-1}=2$, $\sum_s e_s\binom{2k}{2s}=1$. By a certain differencing process we use the binomial coefficient identity above to show that the nth difference $a_r-\binom{n}{1}a_{r+q}+\binom{n}{2}a_{r+2q}-\cdots\pm a_{r+nq}$ is divisible by p^n if $r\geqq 1+[n/2]$. A similar result holds for the nth differences of the integers e_k. The numbers $(-1)^{k-1}a_k$ are positive odd numbers divided by the highest power of 2 that divides k. Values of the first nine a_k are $a_1=1$, $a_2=-1/2$, $a_3=1$, $a_4=-17/4$, $a_5=31$, $a_6=-691/2$, $a_7=43\cdot 127$, $a_8=-257\cdot 3617/8$, $a_9=73\cdot 43867$.

9. *An expansion theorem for a pair of first order equations*, by Professor S. D. Conte, Wayne University.

The principal result of this paper is given by the following theorem. Consider the system of first order equations

$$\begin{aligned} u'(x) - [\lambda a(x) + b(x)]v(x) &= 0, \\ v'(x) + [\lambda c(x) + d(x)]u(x) &= 0, \end{aligned} \tag{1}$$

where b, d are functions of class C' defined over the interval $[0 \leqq x \leqq 1]$, a and c are positive, of class C'' and non-vanishing over this interval, and λ is a parameter, real or complex. The system (1) under the initial conditions

$$u(0) = k_0\alpha, \qquad v(0) = k_0^{-1}\beta, \tag{2}$$

where $k(x) = [a/c]^{1/4}$, $k_0 = k(0)$ and α, β are real constants has the following asymptotic solution for large λ:

$$\begin{aligned} u(x) &= k(x)[\alpha \cos \xi + \beta \sin \xi] + O(1/|\lambda|), \\ v(x) &= k^{-1}(x)[-\alpha \sin \xi + \beta \cos \xi] + O(1/|\lambda|), \end{aligned} \tag{3}$$

where

$$\xi(x) = \lambda h(x) + g(x),$$

$$h(x) = \int_0^x \sqrt{ac}\, dy,$$

$$g(x) = \frac{1}{2}\int_0^x \frac{bc + ad}{\sqrt{ac}}\, dy.$$

The asymptotic solution obtained above is of great value in obtaining expansion theorems for both the regular and singular cases of the system (1).

10. *Open forum: Mathematical and pedagogical notes and devices.*

In this period provided for short unannounced talks, Professor E. E. Moise, Professor G. Y. Rainich, and Professor P. S. Jones respectively contributed comments on the normal form of a plane, a connection between Professor Leisenring's geometrical work and relativity theory, and the "turn-over" top.

At the conclusion of the morning meeting Professor T. H. Hildebrandt commented on the possibilities for subsidized research programs sponsored by the Office of Ordnance Research.

P. S. JONES, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held at William Jewell College, Liberty, Missouri, on April 24, 1953. Professor L. O. Jones, Chairman of the Section, presided at the morning and afternoon sessions.

There were eighty persons in attendance, including the following thirty-four members of the Association:

L. W. Akers, J. J. Andrews, S. Louise Beasley, H. D. Brunk, P. B. Burcham, Mary L. Cummings, C. H. Dalton, W. C. Doyle, Vida E. Dunbar, D. H. Erkiletian, Jr., G. M. Ewing, C. V. Fronabarger, Nola A. Haynes, F. F. Helton, N. Q. Hubbard, G. H. Jamison, L. O. Jones, Nellie

M. Kitchens, W. A. Lafferty, F. H. Lloyd, R. J. Michel, Marie A. Moore, C. V. Newsom, R. M. Rankin, Francis Regan, Lois J. Roper, Marlow Sholander, Sister Ann Elizabeth, Sister M. Pachomia, Sister M. Teresine, J. H. Skelton, W. R. Utz, Margaret F. Willerding, J. L. Zemmer.

At the business meeting the following officers were elected for the coming year: Chairman, Professor G. H. Jamison, Northeast State Teachers College; Vice-Chairman, Professor W. R. Utz, University of Missouri; Secretary-Treasurer, Professor Marie A. Moore, Harris Teachers College; Associate-Secretary, Professor Margaret F. Willerding, Harris Teachers College. It was voted that a committee be appointed to make a study of the requirements of the State Department of Education with respect to mathematics.

The following program was presented:

1. *Symposium on algebra*, Professor W. R. Utz, University of Missouri, Leader.

(i) *Algebra in the high school*, by Superintendent W. F. Williams, Oregon High School.

The speaker feels that algebra can best be taught after a year of general mathematics which stresses fundamental arithmetic. However, if students leave high school with only one unit of mathematics, algebra is to be preferred. For the students who are interested in mathematics and science, a second course in algebra is advised.

(ii) *Is the algebra taught in college really "college algebra,"* by Professor Margaret F. Willerding, Harris Teachers College.

The speaker discussed the content of the three levels of algebra courses available at Harris Teachers and Junior College. Special emphasis was given to the discussion of methods of presentation, including rigor and axiomatic approach, and repetition, both necessary and unnecessary, in the three levels.

(iii) *Remarks concerning mathematics for college freshmen*, by Professor F. F. Helton, Central College.

Some conclusions are given from experiences with college freshmen inadequately prepared for the standard course in college algebra. The so-called intermediate algebra leaves basic difficulties unresolved, since these stem from lack of real understanding of fundamental arithmetic concepts. The mental maturity of college freshmen over high school freshmen permits development of arithmetic processes in terms of their historical growth and the fundamental axioms upon which they are based. Elementary algebra can then be made a natural extension of arithmetic, with repetition and enlarging use of the basic concepts and fundamental assumptions.

(iv) *The algebra program at Washington University*, by Professor Marlow Sholander, Washington University.

The author described the program in algebra offered at Washington University and problems arising in connection with it. He also gave some of the general attitudes towards mathematics and teaching shared by members of the mathematical department.

2. *The undergraduate curriculum in mathematics and its relation to the training of mathematics teachers*, by Dr. C. V. Newsom, New York State Education Department. (By invitation).

This paper is divided essentially into two parts. The first part discusses the author's observa-

tions of some of the serious problems faced by the high schools of this country. The second part considers the ideal undergraduate program for the training of teachers of mathematics. In particular, it is emphasized that greater attention must be given to the interrelationships of mathematics and other aspects of our culture. Specific suggestions are made for the improvement of the curriculum in mathematics. The paper concludes by asserting that the recommended changes in content and emphasis are equally necessary for the program in the liberal arts.

NOLA A. HAYNES, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-seventh annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 18, 1953. Professor E. J. Mickle, Chairman of the Section, presided at the morning and afternoon sessions.

Ninety-eight persons registered in attendance, including the following seventy-four members of the Association:

J. E. Adney, Jr., A. G. Anderson, Grace M. Bareis, I. A. Barnett, H. M. Beatty, G. M. Bloom, Foster Brooks, O. E. Brown, V. B. Caris, Dorothy I. Carpenter, M. N. Chase, C. C. Crell, R. C. Davis, B. B. Dressler, Wade Ellis, P. L. Evans, H. E. Fettis, D. T. Finkbeiner, A. L. Fisch, M. P. Fobes, H. W. Godderz, L. J. Green, Marshall Hall, Jr., H. G. Harp, R. G. Helsel, G. H. Hembree, Carl Holtom, R. Y. Iwanchuk, M. L. Johnson, Margaret E. Jones, Chosaburo Kato, D. M. Krabill, D. H. Kraft, Nathan Lazar, F. C. Leone, H. D. Lipsich, L. L. Lowenstein, Beckham Martin, Margaret E. Mauch, S. W. McCuskey, E. J. Mickle, L. H. Miller, Knox Millsaps, Max Morris, J. R. Musselman, Helen Olney, W. R. Orton, P. M. Pepper, H. S. Pollard, Tibor Rado, R. F. Reeves, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, L. V. Robinson, S. A. Rowland, C. E. Sealander, Samuel Selby, L. L. Shetler, R. L. Shively, Ruth B. Smyth, R. E. Thomas, Michael Tikson, H. S. Toney, W. R. Transue, E. P. Vance, E. H. Wang, D. R. Whitney, R. B. Wilkermuth, C. O. Williamson, J. A. Wilson, E. H. Wohler, Alberta Wolfe, A. D. Ziebur.

The following officers were elected for the coming year: Chairman, Professor P. R. Rider, Wright-Patterson Air Force Base; Secretary-Treasurer, Professor Foster Brooks, Kent State University; third member of the Executive Committee, Professor H. H. Alden, Ohio State University; Program Committee: Chairman, Professor L. L. Lowenstein, Kent State University; Professor Marion D. Wetzel, Denison University; Professor Wade Ellis, Oberlin College.

Professor S. A. Rowland of Ohio Wesleyan University presented the following resolution which the Section adopted, directing that copies be sent to the Governor of Ohio and to the chairmen of the Committees on Education of the Senate and the House of Representatives of the Ohio Legislature:

"The members of the Ohio Section of the Mathematical Association of America present at their annual meeting at Columbus, Ohio, April 18, 1953, note with deep interest and approval that a bill has been introduced into the Ohio Legislature providing for a commission to study the Ohio educational system and to make recommendations for its improvement.

"We, as teachers of mathematics in the universities and colleges of Ohio and as practicing mathematicians, are aware of the great task confronting the public

schools, whose teachers must try to impart the foundations of good citizenship to all students regardless of the differences in ability, aptitude, and ambition. But as teachers we are continually observing the great disparity in the preparation of high school students for college entrance even among students who wish to enroll in the same curriculum in college.

"We therefore believe that one very important aspect of the commission's work should be to cause to be established standards of achievement to be attained by high school students as pre-requisite to certification for college entrance. Such a program to be successful would probably require uniform college entrance examinations in such fundamental subjects as English, mathematics, and history, and special examinations in other subjects requiring special prerequisites, such as engineering, agriculture, science, and languages.

"We believe that the large amount of money now being spent on our total program of education from kindergarten to Ph.D. demands that the transition from school to college be efficiently coordinated. We earnestly request that this matter be given full consideration."

The following papers were presented:

1. *Combinatorial problems*, by Professor H. B. Mann, Ohio State University. (By invitation).

An incomplete balanced block design is an arrangement of v varieties into b blocks of k different varieties each such that:

(1) Every variety occurs the same number r of times,

(2) Each pair of varieties occurs the same number λ of times in a block.

A large number of such designs have been found, notably by the Indian mathematicians under the leadership of R. C. Bose. However, the problem of finding necessary and sufficient conditions for the existence of solutions with given v, b, r, k, λ is still far from its solution. It is easy to see that we must have $vr=bk$, and $r(k-1)=\lambda(v-1)$. Significant advances have recently been made and some non-trivial necessary conditions were found by Bruck, Ryser, Chowla, and Schutzenberger. A closely related problem is the problem of the existence of difference sets. A set of distinct residues $a_1, \cdots, a_k$ mod v is called a difference set of multiplicity λ if the congruence $a_i-a_k=d(v)$ has exactly λ solutions for every $d\not\equiv 0$. It is known that such a set always exists if $\lambda=1$ and $(k-1)$ is a prime power. This problem was opened by a significant discovery due to Marshall Hall. Hall's theorem together with some due to the author enabled the author and T. A. Evans to show that no difference set with $\lambda=1$, $k-1\leqq 1600$ exists unless $k-1$ is the power of a prime.

2. *A system of simultaneous quadratic equations arising from a problem in electrical resistances*, by Professor I. A. Barnett, University of Cincinnati.

The equations in question are $y_i^2=(y_i-k_i)\sum_{j=1}^{n} k_j$, $(i=1, \cdots, n)$. For the case $n=3$, necessary and sufficient conditions are given in order that the solutions y_i be integral. The general case is reduced to the equation in c, $\sum_{j=1}^{n} \epsilon_j\sqrt{1-ck_i}=0$, where the ϵ_j take on the values ± 1. Finally, the problem of a wire of variable diameter is considered.

3. *A generalization of a theorem of Ankeny and Rogers*, by Professor H. B. Mann, Ohio State University.

Let F denote an algebraic number field, R the field of rational numbers and ξ_t a primitive tth root of unity. Further let $m=2^k m'$ where m' is odd and let α be the largest integer such that

$\xi_2\alpha+\xi^2\alpha^{-1}\epsilon F$. The following result is proved:

If $a\epsilon F$ is an mth residue for a set of prime ideals of Dirichlet density 1, then

(1) The number a is an mth power if the intersection $F\bigcap R(\xi_2{}^k)$ is not real or if $\alpha\leqq k$;

(2) The number a is either an mth power or

$$a=\left(\frac{4}{\xi_2\alpha+\xi_2\alpha^{-1}+2}\right)^{m/2}s^m,$$

$s\epsilon F$ if $\alpha<k$ and $F\bigcap R(\xi_2{}^k)$ is real.

Moreover, if F satisfies the conditions of case (2), then

$$\left(\frac{4}{\xi_2\alpha+\xi_2\alpha^{-1}+2}\right)^{m/2}$$

is m-th residue for every prime ideal p but is not an mth power.

4. *Mutation geometry*, by Dr. Beckham Martin, Toledo.

The author stated that the essentials of mutation geometry are contained in one postulate, called alpha, and a proposition, called omega. Alpha requires prototype products of the form $a\cdot b$ to be tempo-locally invariant. Omega is a proposition of mutation, splintering omega products of the form $(a\cdot c)(b\cdot c)$ into the sum of two prototypes:

$$(a\cdot b)(b\cdot c)=(a\cdot b+b\cdot d)/2, \quad \text{where}$$

b is called the mutate of a. Mutates splintered from two or more omega products are communalized to a chosen primordial prototype by alpha. Vectors in the composite primordial prototype are called migrates. Illustrative examples from analytic, college, projective, and differential geometry were solved.

5. *A complex exponential shift to solve some differential equations*, by Professor H. W. Godderz, Ohio Northern University.

The exponential shift $\Phi(D)e^{ax}X=e^{ax}\Phi(D+a)X$ was extended to show $\Phi(D)e^{ibx}X=e^{ibx}\Phi(D+ib)X$. A linear differential equation with constant coefficients whose right hand member contains $\sin x$ or $\cos x$ can be written in complex form, transformed by Euler's relation and then operated on by the complex exponential shift. This saves a great deal of differentiation in the method of undetermined coefficients if the right hand member is of the form $e^x\sin x$ and especially when the complementary function contains the right hand member.

6. *On the continuous solutions of a certain functional equation*, by Dr. R. F. Reeves, Ohio State University.

In a recent issue of this MONTHLY D. J. Newman posed the question, "Is there a function $f(x)$ continuous in the closed interval (0, 1) and such that in this interval $f(x)+f(x^2)=x$?" The answer is no. There exists a unique function which satisfies this equation for $0\leqq x<1$ and which is continuous at $x=0$. Similarly, there exists a unique function which satisfies the equation for $0<x\leqq 1$ and which is continuous at $x=1$. The fact that these two solutions prove to be distinct justifies the negative answer. This result is extended to a more general equation, and further information is obtained about the solutions of this equation.

7. Chairman's address: *Mappings and measures*, by Professor E. J. Mickle, Ohio State University.

Let T be a continuous mapping from a square Q in the uv-plane into xyz-space and let $H^2(E)$ denote the Hausdorff 2-dimensional measure of a set E in xyz-space. The talk discussed the choice of a suitable counting function $N(x, y, z; T)$ so that for the Lebesgue area $A(T)$ of the mapping T the equality $A(T)=\int N(x, y, z; T)dH^2$ holds.

8. *Generalizing the calculus*, by Dr. L. V. Robinson, Wright-Patterson Air Base.

An operator,

$$\lambda = P_1 \frac{\partial}{\partial x_1} + P_2 \frac{\partial}{\partial x_2} + \cdots + P_n \frac{\partial}{\partial x_n},$$

where P_k $(k=1, 2, \cdots, n)$ are differentiable functions of the independent variables x_i, is defined. A generalized calculus in which the operators are λ and λ^{-1} is then compared to the ordinary calculus in which the operators are $D=d/dx$ and its inverse D^{-1}. The advantages are demonstrated in solving partial differential equations.

9. *Circular probability paper*, by Professor F. C. Leone, Case Institute of Technology, and Professor C. W. Topp, Fenn College, presented by Professor Leone.

This paper describes circular probability paper, its construction and use, and several examples in which it is used to compute probabilities. The illustrations include the historical case for which the paper was designed by H. H. Germond, the case of the volume above an irregular region, the bivariate and uncorrelated case and the bivariate and correlated case. Circular probability paper can be of great assistance in solving some industrial problems in probability which were prohibitive because of the extensive calculation involved.

10. *The Air Force's newest large scale computer—OARAC*, by Mr. R. W. House, Wright-Patterson Air Base, introduced by Mr. D. H. Kraft.

The OARAC (Office of Air Research Automatic Computer) is a single-address, general-purpose computer with magnetic drum storage for 10,000 ten-digit numbers plus sign. The input-output medium is magnetic tape. The speed of operation is about 5,000 operations per minute. Sub-routines for computing the square root, sine and cosine, logarithm, arctangent, and exponential are stored in the memory. The average computing in seconds for each is 0.6, 0.8, 1.0, 1.2, and 1.2 respectively with a ten-place evaluation for a ten-place argument. The computer is located at Wright-Patterson Air Force Base, Ohio.

11. *Numerical methods in matrix theory*, by Mr. Michael Tikson, Wright-Patterson Air Base.

The speaker discussed the application of numerical matrix methods to automatic computing machines. These applications were illustrated with a problem arising from the analysis of a propeller blade. Computational checks, the characterization of ill-conditioned matrices, a variation of the Gauss elimination process for inversion and the Power method for determining dominant eigenvalues were mentioned.

12. *Error analyses of certain sets of non-homogeneous simultaneous linear algebraic equations*, by Mr. D. H. Kraft, Wright-Patterson Air Force Base.

For a set of non-homogeneous linear algebraic equations $AX=B$ a procedure was presented for obtaining an estimate of errors which result only from those calculations performed on the n elements of the coefficient matrix in solving the system by a direct method, such as Gauss's elimination or the Crout method. These errors are: (a) rounding errors and (b) errors arising from loss of significant figures in the subtraction of nearly equal numbers. The procedure is to solve $AY=C$, where each element of the column of vector C is the sum of the elements of the corresponding row of A. This system has the exact solution $Y_1=\cdots=Y_n=1$. If errors contributed by B can be neglected, then the number of digits in the numerically computed Y_i that are in agreement with the exact solution $Y_i=1$ is a measure of the number of significant digits in the solution of $AX=B$.

13. *Training of personnel in fields related to machine computation*, by Professor H. S. Toney, Wright-Patterson Air Force Base, Ohio.

To provide needed personnel trained for the machine solution of engineering problems the Computation Branch, Flight Research Laboratory at Wright Air Development Center has conducted two classes of a ten-week training course in the theory and methods of machine computation, the logical design of analog and digital computers, and coding techniques for the REAC, the IBM's CPC, and the OARAC. The students were airmen with degrees in mathematics, science, or engineering. The speaker described the course content and management. His faculty consisted of specialists chosen from the regular staff of the Computation Branch.

14. *On grading*, by Professor A. G. Anderson, Oberlin College.

A method is presented for removing some of the arbitrariness involved in the process of assigning grades to students. It is not intended to replace the "rule of thumb" method for those who have taught for some time but rather to provide the new instructor with a flexible method which allows him to incorporate his own ideas on the subject while conforming to any general policy set up by his institution. The key to the method is the use of "theoretical percentiles" within a population rather than indulging in the unfortunate practice of assigning predetermined fixed percentages of each possible grade within a given class (sometimes erroneously referred to as "grading on the curve").

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA

IOWA, Iowa State College, Ames, April, 1954.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 5, 1953.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, April, 1954.

MINNESOTA, Bemidji State Teachers College, October 10, 1953.

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA

OHIO, April, 1954.

OKLAHOMA, Oklahoma City University, October 30, 1953.

PACIFIC NORTHWEST, Reed College, Portland, Ore., June 18, 1954.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe.

TEXAS, Texas Technological College, Lubbock, April, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.

WADE: CALCULUS

By Thomas L. Wade, Florida State University

CARIS: ELEMENTS OF ALGEBRA

By Vaughn B. Caris, Ohio State University

ZANT: COLLEGE ALGEBRA AND PLANE TRIGONOMETRY

By James H. Zant, Oklahoma Agricultural and Mechanical College

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